§5.1 (Precise Def of) Limits of Functions

Today: \( \lim_{x \to a} f(x) \)

Why didn't we start with this, instead of \( \lim_{n \to \infty} s_n \)?

(1) Teaching: sequence def™ not as complicated

(2) Textbook def™ more complicated than needed.

You can use book as a reference, but anything you need to know for final will be covered here.

Wednesday: limit wrapup, prep for final exam.
From the beginning, "limits" and "for getting really, really close to L" were derided.

Berkeley (18th Century)

infinitesimals are the "ghosts of vanishing quantities."

Russell (20th Century)

... are "unnecessary, erroneous and self-contradictory"
Leibniz

Newton

Cauchy

Weierstrass
With sequences:

\[ \lim_{n \to \infty} S_n = L \text{ if } \forall \varepsilon > 0, \exists N \text{ s.t. } n > N \text{ forces } |S_n - L| < \varepsilon \]

i.e. eventually, every # in seq. is (arbitrarily) close to \( L \).
More generally, for \( f: \mathbb{R} \to \mathbb{R} \),

\[
\lim_{x \to a} f(x) = L \text{ means }
\]

As \( x \) approaches \( a \), \( f(x) \) gets closer and closer to \( L \).

**Precise Def:** \( \lim_{x \to a} f(x) = L \) means:

\[
\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.
\]

\( x \) close to \( a \), but not equal to \( a \).  \( f(x) \) close to \( L \) (arbitrarily close to \( L \) for \( \forall \varepsilon > 0 \)).
\[ \lim_{x \to a} f(x) = L \text{ if: } \forall \varepsilon > 0 \exists \delta \text{ s.t. } 0 < |x-a| < \delta \text{ forces } |f(x) - L| < \varepsilon. \]

⚠️ As with sequences, ORDER IS IMPORTANT.

\( \varepsilon \) is chosen first. Then you have to find \( \delta \) to make def\(^\text{\#}\) work.

As with limits of sequences, we'll use a "Think" step and a "Proof" step.
Ex. Prove $f(x) = 3x - 1$ is continuous at $x=2$.

Need to show: $\lim_{x \to 2} f(x) = f(2)$, i.e. $\lim_{x \to 2} 3x - 1 = 5$.

Think: Want to find $\delta$ s.t. $|x-2| < \delta \Rightarrow |3x-1-5| < 3$.

**Method 1**

\[
\begin{align*}
|3x-6| &< 3 \\
-3 &< 3x - 6 < 3 \\
-3 &< 3(x-2) < 3 \\
-\frac{1}{3} &< x-2 < \frac{1}{3} \\
|x-2| &< \frac{1}{3}
\end{align*}
\]

**Method 2**

\[
\begin{align*}
|3x-6| &< 3 \\
|3(x-2)| &< 3 \\
3 \cdot |x-2| &< 3 \\
|x-2| &< \frac{3}{3} \Rightarrow |x-2| < \frac{1}{3}
\end{align*}
\]
**Ex.** Prove \( f(x) = 3x - 1 \) is continuous at \( x = 2 \).

Need to show: \( \lim_{{x \to 2}} f(x) = f(2) \), i.e. \( \lim_{{x \to 2}} 3x - 1 = 5 \).

**Proof:**

Let \( \varepsilon > 0 \) and choose \( \delta = \frac{\varepsilon}{3} \). Then \( 0 < |x - 2| < \delta \Rightarrow 

\[
| (3x - 1) - 5 | = | 3x - 6 | = 3 |x - 2| < 3 \left( \frac{\varepsilon}{3} \right) = \varepsilon
\]

\[
\forall \varepsilon > 0 \exists \delta \text{ s.t. } 0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon
\]
Ex: Prove \( \lim_{{x \to 0}} x^4 = 0 \)

T: Want to find \( \delta \) s.t. \( 0 < |x - 0| < \delta \) forces \( |x^4 - 0| < \varepsilon \).

\[
|x^4| = |x|^4 = |x-0|^4 < \varepsilon \quad \Rightarrow \quad |x-0| < \varepsilon^{1/4}
\]

P: Let \( \varepsilon > 0 \). Choose \( \delta = 4\sqrt[4]{\varepsilon} = 2\varepsilon^{1/4} \). Then \( |x-0| < \delta \),

\[
|x^4 - 0| = |x^4| = |x|^4 = |x-0|^4 < (\varepsilon^{1/4})^4 = \varepsilon
\]

\[\forall \varepsilon > 0 \exists \delta \text{ s.t. } 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon \]
Ex. Prove \( \frac{2}{x-4} \) \( x^2 - 2x - 3 = 5 \)

Think: want \( 0 < |x-4| < \delta \) to force \( |(x^2 - 2x - 3) - 5| < \varepsilon \).

We want \( |x^2 - 2x - 8| = |(x+2)(x-4)| = |x+2| \cdot |x-4| < \varepsilon \).

\( \square \) We can make \( |x-4| \) small, but not \( |x+2| \).... or can we?!

If \( |x-4| \) really small, then \( x \approx 4 \) and \( x+2 \approx 6 \).

If I eventually make sure \( \delta < 1 \), then

\[
|x-4| < \delta = 1 \\
-1 < x-4 < 1 \\
3 < x < 5 \\
5 < x+2 < 7 \\
|x+2| < 7
\]
So if \(|x-4|<1\), \(|x^2-2x-8| = |x+2| \cdot |x-4| < 7 \cdot |x-4| < 3\)

suggests \(|x-4| < \frac{3}{7} = \delta\).

**Proof:** Let \(\varepsilon > 0\), and choose \(\delta = \min\{1, \frac{\varepsilon}{7}\}\). Thus if \(0 < |x-4| < \delta\) then not only is \(|x-4| < \frac{3}{7}\), but also \(|x-4| < 1\)

\[-1 < x-4 < 1\]

\(|x+2| < 7\).

Furthermore,

\(|(x^2-2x-3)-5| = |x^2-2x-8| = |x+2| \cdot |x-4| < 7 \cdot \left(\frac{3}{7}\right) = 3\).
Ex Prove $\lim_{x \to -2} x^2 - 1 = 3$.

\[
| (x^2 - 1) - 3 | = | x^2 - 4 | = | x - 2 | \cdot | x + 2 |.
\]

We can make $| x + 2 |$ small; if $| x + 2 | < 1$, then

\[
-1 < x + 2 < 1
\]

\[
-5 < x - 2 < -3
\]

\[
| x - 2 | < 5
\]

Then $| x - 2 | \cdot | x + 2 | < 5 | x + 2 | < 3$ suggest $\delta = 3/5$. 
Let $\varepsilon > 0$.

Proof Let $\varepsilon > 0$. Choose $\delta = \min \{1, 3/5\}$. Then $0 < |x+2| < \delta$ means $|x+2| < 3/5$ and $|x+2| < 1$; the latter forces

\[-1 < x+2 < 1\]
\[-5 < x-2 < -3\]
\[|x-2| < 5\]

Thus:

\[|(x^2-1)-3| = |x^2-4| = |x-2| \cdot |x+2| < 5 \cdot (3/5) = 3.\]
Final Exam

* Three past finals (with solutions) that will be posted online

Review guide posted today/tomorrow.