§ 8.3 Power Series

So far our series have been infinite sum of preselected numbers. Given \((a_n) = (a_1, a_2, a_3, \ldots)\) we analyze

\[ \sum a_n = a_1 + a_2 + a_3 + a_4 + \ldots \]

In this section, our series are functions which depend on a variable. Two issues:

1. When does make sense?

2. Why would we care?
**Def** Let $\alpha_n$ be a sequence. Then
\[ \sum_{n=0}^{\infty} \alpha_n x^n = a_0 x^0 + a_1 x + a_2 x^2 + \ldots \]
is a power series. $\alpha_n$ is coeff of $x^n$. (the $n^{th}$ coeff)

**Notes**

For a specific $x$, we get a regular old series
\[ \sum_{n=0}^{\infty} \frac{1}{n+1} x^n = 1 + \frac{1}{2} x + \frac{1}{3} x^2 + \frac{1}{4} x^3 + \ldots \]

For $x = 1$,
\[ \sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots = +\infty \]

For $x = \frac{1}{2}$,
\[ \sum_{n=0}^{\infty} \frac{1}{n+1} (\frac{1}{2})^n = 1 + \frac{1}{4} + \frac{1}{12} + \frac{1}{32} + \ldots \]

Converges by comparison test? \( \frac{1}{n+1} (\frac{1}{2})^n \leq (\frac{1}{2})^n \)

**Main goal of this section:** Simultaneously find all values of $x$ for which $\sum \alpha_n x^n$ converges.
2. **Why** do all of this?

It gives us another way to represent functions.

**Ex.** If $x \in (-1, 1)$ \[so \ |x| < 1\]

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots
\]

nicer than

**Ex.** $\forall x \in \mathbb{R}$, it turns out

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots
\]

still nicer than

3. **Advantages** to power series form. (Sometimes.)
\[ e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24} + \ldots \]

Set \( u = x^2 \) to get

\[ e^{x^2} = 1 + (x^2) + \frac{(x^2)^2}{2} + \ldots = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \ldots \]

\[ \int e^{x^2} \, dx \text{ cannot be written using "elementary" fns, but in Advance Calc / Real Analysis we prove you can integrate a power series term by term:} \]

\[ \int e^{x^2} \, dx = \left( x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \ldots \right) + C \]
Another way power series arise is through **Taylor Polynomials**.

\( e^x \) is hard to compute, but polynomials are "easy," esp. for a computer...

Can we find poly's of degree \( n \), \( p_n(x) \approx e^x \) near \( x=0 \)?
(Match fn value and as many derivatives as possible.)

\[
\begin{align*}
p_0(x) &= 1 & p_0(0) &= 1 = e^0 & \checkmark \\
p_1(x) &= 1 + x & p_1(0) &= 1 + 0 = 1 = e^0 & \checkmark \\
\end{align*}
\]

\[ p_1'(0) = 1 = 1^{st} \text{ deriv of } e^x @ x=0 \checkmark \]
In general, $p_a(x) = 1 + x + \frac{x^2}{2}$.
Main Goal: For what values of \( x \) does \( \sum a_n x^n \) converge?

Think: what's the domain?

\[
\sum_{n=1}^{\infty} \left( \frac{a^n}{n!} \right) x^n = 2x + 2x^2 + \frac{8}{3}x^3 + \ldots
\]

For some chosen \( x \), use ratio test:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n \to \infty} \left| \frac{2x \cdot x}{2^n} \right| = 2|x| < 1
\]

if \( |x| < \frac{1}{2} \), i.e. series converges for \( x \in (-\frac{1}{2}, \frac{1}{2}) \)

and if \( |x| > \frac{1}{2} \), then \( \lim \) is \( >1 \) and series diverges.
We need to check cases where limit is 1 by hand.

i.e. \(21x1 = 1\), \(x=\frac{1}{2}\) or \(x=-\frac{1}{2}\)

\[x = \frac{1}{2}: \sum \left(\frac{3^n}{n}\right) = \frac{1}{n} + \infty \text{ diverges}\]

\[x = -\frac{1}{2}: \sum \left(\frac{(-3/2)^n}{n}\right) = \sum (-1)^n \cdot \frac{1}{n} \text{ converges (alt. harm. series)}\]

Thus series converges iff \(x \in [-\frac{1}{2}, \frac{1}{2}]\).

Interval of converge is \(\frac{r}{2}\)

Radius of converge is \(r = \frac{1}{2}\)