Math 3283W
Fall 2010
Final Exam
December 21, 2010
Time Limit: 120 minutes

Name (Print): ______________________
Student ID: ______________________
Section Number: ______
Teaching Assistant: ______________________
Signature: ______________________

This exam contains 10 numbered problems. Check to see if any pages are missing. Point values are in parentheses. No books, notes, or electronic devices are allowed.

1	20 pts	
2	20 pts	
3	20 pts	
4	20 pts	
5	20 pts	
6	20 pts	
7	20 pts	
8	20 pts	
9	20 pts	
10	20 pts	
TOTAL	200 pts	

1. (20 points) (5 points each) Statements.

   a. State the Completeness Axiom.

   b. State the Bolzano-Weierstrass Theorem.

   c. State the Monotone Convergence Theorem.

   d. State the Ratio Test, proven in this class, concerning convergence of series. (Omit the third of the three statements that describes when the ratio test is inconclusive.)

2. (20 points) (5 points each) Definitions. Complete each sentence.

a. A sequence (a_n) *converges* to a if ...

b. A series $\sum_{n=1}^{\infty} a_n$ *converges* to s if ...

c. Given a function $f : D \to \mathbb{R}$, and an accumulation point c of the domain D, we say that

$$\lim_{x \to c} f(x) = L$$

if ...

d. The real number s is the *limit superior* of a bounded sequence (a_n) (that is, $s = \limsup a_n$) if ...

3. (20 points) (5 points each) Calculations. No justification necessary.

a. Find the closure of the set

$$A = \bigcap_{n=1}^{\infty} (n, n+1).$$

b. Find the set of subsequential limits of the sequence $(\frac{1}{1}, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \ldots)$.

c. Find the sum of the convergent series

$$\sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^n.$$

(Note the index of the first term of the series.)

d. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^n.$$

4. (20 points) (5 points each) Examples. No justification necessary.

   a. Give an example of a series that is convergent but not absolutely convergent.

   b. Give an example of a divergent p-series. That is, choose a p that makes the corresponding p-series divergent, and write the series in sum notation.

   c. Give an example of a sequence of irrational numbers that converges to a rational number.

   d. Give an example of an infinite collection $A_1, A_2, \ldots$ of open subsets of $\mathbb{R}$ with the property that

$$\bigcap_{n=1}^{\infty} A_n$$

   is not open.

5. (20 points) (20 points) Let A and B be sets, and let $f : A \to B$ be a function. Suppose that B_1 and B_2 are subsets of B. Prove that

$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2).$$

6. (20 points) a. (5 points) Is $\mathbb{Q} \cap (0, 1)$ a countable set? Justify your answer directly from the definition of *countable*.

b. (15 points) Is $\mathbb{Q} \cap (0, 1)$ a compact set? Justify your answer directly from the definition of *compact*.

7. (20 points) a. (15 points) Prove that, for all $n \geq 2$, we have

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

b. (5 points) Let us say that the *infinite product*

$$\prod_{i=k}^{\infty} a_i$$

has value P if the sequence (s_n) of partial products

$$s_n = \prod_{i=k}^{n} a_i = a_k \cdot a_{k+1} \cdot \cdots \cdot a_n$$

converges to P. Find the value of the infinite product

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right).$$

8. (20 points) (10 points each) Determine whether the following series converge. Show your work. If the series converges, find its sum.

a.

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

b.

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)}$$

9. (20 points) Find the set of all real numbers x for which the following series converges:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}.$$

Show your work.

10. (20 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions that are continuous at $a \in \mathbb{R}$. Prove that the function $f + g$ is continuous at a.