This exam contains 10 numbered problems. Check to see if any pages are missing. Point values are in parentheses. No books, notes, or electronic devices are allowed.

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1. (20 points) (5 points each) Statements.
   a. State the Completeness Axiom.
      \[ \text{Every nonempty subset of real numbers} \]
      \[ \text{that is bounded above has a supremum in } \mathbb{R}. \]
   b. State the Bolzano-Weierstrass Theorem.
      \[ \text{Bounded, infinite subsets of } \mathbb{R} \]
      \[ \text{have at least one accumulation pt.} \]
   c. State the Monotone Convergence Theorem.
      \[ \text{A monotone sequence} \]
      \[ \text{converges if and only if it is bounded.} \]
   d. State the Ratio Test, proven in this class, concerning convergence of series. (Omit the third of the three statements that describes when the ratio test is inconclusive.)
      1) If \( \lim \sup \left| \frac{a_{n+1}}{a_n} \right| < 1 \), then \( \sum a_n \) converges.
      2) If \( \lim \inf \left| \frac{a_{n+1}}{a_n} \right| > 1 \), then \( \sum a_n \) diverges.
      
      \[ \text{not sequences, not power series} \]
2. (20 points) (5 points each) Definitions. Complete each sentence.

a. A sequence \((a_n)\) converges to \(a\) if …
\[
\forall \varepsilon > 0, \exists N \text{ s.t. } n > N \implies |a_n - a| < \varepsilon
\]

b. A series \(\sum_{n=1}^{\infty} a_n\) converges to \(s\) if …
the sequence \(s_1 = a_1, \ldots + a_n\) of partial sums converges to \(s\) as above.

\[
\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } x \in D \text{ and } 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon
\]

\[
\lim_{x \to c} f(x) = L
\]

d. The real number \(s\) is the limit superior of a bounded sequence \((a_n)\) (that is, \(s = \limsup a_n\)) if …
it is the supremum of the set of subsequential limits of \((a_n)\).
3. (20 points) (5 points each) Calculations. No justification necessary.

a. Find the closure of the set

\[ A = \bigcap_{n=1}^{\infty} (n, n+1). \]

\[ \emptyset \]

b. Find the set of subsequential limits of the sequence \((\frac{1}{1}, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \ldots)\).

\[ [0, 1] \]

c. Find the sum of the convergent series

\[ \sum_{n=2}^{\infty} \left( -\frac{1}{2} \right)^n. \]

(Note the index of the first term of the series.)

\[ \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n = \frac{1}{1-\left( -\frac{1}{2} \right)} = \frac{2}{3} \]

\[ \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n = \frac{2}{3} - \left( 1 - \frac{1}{2} \right) = \frac{1}{6} \]

3 pts for this or anything using \[ \frac{1}{1-r} \].

d. Find the radius of convergence of the power series

\[ R = \lim_{n \to \infty} \frac{(2n)!}{(n!)^2} \cdot \frac{(n+1)!}{(2(n+1))!} = \lim_{n \to \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} \]
4. (20 points) (5 points each) Examples. No justification necessary. **ALL OR NOTHING**.

a. Give an example of a series that is convergent but not absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

b. Give an example of a divergent p-series. That is, choose a $p$ that makes the corresponding p-series divergent, and write the series in sum notation.

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

or any other $p \leq 1$.

c. Give an example of a sequence of irrational numbers that converges to a rational number.

$$a_n = \frac{\sqrt{2}}{n}, \quad (\to 0)$$

d. Give an example of an infinite collection $A_1, A_2, \ldots$ of open subsets of $\mathbb{R}$ with the property that

$$\bigcap_{n=1}^{\infty} A_n$$

is not open.

$$A_n = \left( \frac{-1}{n}, \frac{1}{n} \right), \quad n \geq 1.$$
5. (20 points) (20 points) Let $A$ and $B$ be sets, and let $f : A \to B$ be a function. Suppose that $B_1$ and $B_2$ are subsets of $B$. Prove that

$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2).$$

**Proof:**

If $f(a) \in B_1$, then $a \in f^{-1}(B_1)$ and hence $a \in f^{-1}(B_1) \cup f^{-1}(B_2)$.

Else $f(a) \in B_2$. Then $a \in f^{-1}(B_2)$ and hence $a \in f^{-1}(B_1) \cup f^{-1}(B_2)$.

**Scores:**

- Structure of proving set equality: 5
- Correct understanding of preimage: 5
- Correct understanding of union: 5
- Discretionary, form/notation: 3
6. (20 points) a. (5 points) Is \( \mathbb{Q} \cap (0,1) \) a countable set? Justify your answer directly from the definition of countable.

1. Yes. We exhibit a bijection \( f : \mathbb{N} \to \mathbb{Q} \cap (0,1) \) (that is, a sequence that is 1-1 and onto)

\[
\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \ldots
\]

2. lowest terms listed only \( \Rightarrow 1-1 \).

b. (15 points) Is \( \mathbb{Q} \cap (0,1) \) a compact set? Justify your answer directly from the definition of compact.

5. No: there is an infinite cover of \( \mathbb{Q} \cap (0,1) \) with no finite subcover:

\[ \{ A_n : n \geq 2 \} \text{ where } A_n = \left( \frac{1}{n}, 1 - \frac{1}{n} \right) \]

For any finite subcollection, let \( N \) be the largest index of all the included sets.

Then \( \frac{1}{N} \in \mathbb{Q} \cap (0,1) \) but not in the union of the finite subcollection. (Or, \( \exists q \in \mathbb{Q} \cap (0, \frac{1}{N}) \), since \( \mathbb{Q} \) is dense in \( \mathbb{R} \).)
7. (20 points) a. (15 points) Prove that, for all \( n \geq 2 \), we have
\[
(1 - \frac{1}{2^2}) (1 - \frac{1}{3^2}) (1 - \frac{1}{4^2}) \cdots (1 - \frac{1}{n^2}) = \frac{n+1}{2n}.
\]

By induction:

(3) When \( n = 2 \), we have \( (1 - \frac{1}{2^2}) = \frac{3}{4} = \frac{2+1}{2 \cdot 2} \).

Suppose the statement is true for \( n = k \), and show it is true for \( n = k+1 \).

\[
(1 - \frac{1}{2^2}) (1 - \frac{1}{3^2}) \cdots (1 - \frac{1}{k^2}) (1 - \frac{1}{(k+1)^2}) = \frac{k+1}{2k} (1 - \frac{1}{(k+1)^2})
\]

\[
= \frac{k+1}{2k} \cdot \frac{k^2+2k+1 - 1}{(k+1)^2} = \frac{(k+1) \cdot k(k+2)}{2k(k+1)^2}
\]

\[
= \frac{k+2}{2(k+1)} = \frac{(k+1) + 1}{2(k+1)}.
\]

b. (5 points) Let us say that the infinite product
\[
\prod_{i=k}^{\infty} a_i
\]

has value \( P \) if the sequence \( (s_n) \) of partial products
\[
s_n = \prod_{i=k}^{n} a_i = a_k \cdot a_{k+1} \cdots a_n
\]

converges to \( P \). Find the value of the infinite product
\[
\lim_{n \to \infty} \frac{n+1}{2n}
\]

all or nothing.
8. (20 points) (10 points each) Determine whether the following series converge. Show your work. If the series converges, find its sum.

a. \[ \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \] (partial fractions)

\[ S_n = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots \]

\[ = \frac{1}{2} - \frac{1}{n+2} \rightarrow \frac{1}{2} \text{ converges to } \frac{1}{2}. \]

b. \[ \sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)} \]

1. \[ 2n^2 > n^2 + 3n + 2, \text{ and hence } \]

2. \[ \frac{n}{2n^2} < \frac{n}{n^2 + 3n + 2} \]

3. \[ \sum_{n=1}^{\infty} \frac{1}{2n} \text{ diverges (for if it converged, so would } \sum_{n=1}^{\infty} \frac{1}{n} \text{)} } \]

4. By comparison test, \[ \sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)} \text{ diverges. } \]
9. (20 points) Find the set of all real numbers $x$ for which the following series converges:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}.$$ 

Show your work.

$${a_n} = \frac{1}{n}$$

(5) $R = \lim \frac{a_n}{a_{n+1}} = \lim \frac{n+1}{n} = 1.$

(5) When $x = 3$, the series is $\sum \frac{1}{n}$, which diverges.

(5) When $x = 6$, the series is $\sum \frac{(-1)^n}{n}$, which converges.

(5) The series converges when $x \in [1, 3)$. 
10. (20 points) Suppose that \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) are functions that are continuous at \( a \in \mathbb{R} \). Prove that the function \( f + g \) is continuous at \( a \).

\( \text{(directly, not using sequences)} \)

(2) Let \( \varepsilon > 0 \) be given.

(2) \( f \) is at \( a \): \( \exists \delta_1 > 0 \) s.t. if \( |x-a| < \delta_1 \),

then \( |f(x) - f(a)| < \frac{\varepsilon}{2} \).

(2) \( g \) is at \( a \): \( \exists \delta_2 > 0 \) s.t. if \( |x-a| < \delta_2 \),

then \( |g(x) - g(a)| < \frac{\varepsilon}{2} \).

(4) Let \( \delta = \min \{ \delta_1, \delta_2 \} \).

(2) If \( |x-a| < \delta \), then \( \varepsilon \).

\( 2 \) \[ |(f+g)(x) - (f+g)(a)| \]

\[ = |f(x) - f(a) + g(x) - g(a)| \]

\[ \leq |f(x) - f(a)| + |g(x) - g(a)| \]

\[ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]