Key / Grading Guide

Math 3283W
Fall 2010
Final Exam
December 21, 2010
Time Limit: 120 minutes

Name (Print): ______________________
Student ID: ______________________
Section Number: ______
Teaching Assistant: ______________________
Signature: ______________________

This exam contains 10 numbered problems. Check to see if any pages are missing. Point values are in parentheses. No books, notes, or electronic devices are allowed.

1	20 pts	
2	20 pts	
3	20 pts	
4	20 pts	
5	20 pts	
6	20 pts	
7	20 pts	
8	20 pts	
9	20 pts	
10	20 pts	
TOTAL	200 pts	

1. (20 points) (5 points each) Statements.

a. State the Completeness Axiom.

1

Every nonempty subset of real numbers that is bounded above has a supremum in $\mathbb{R}$.

2 2

b. State the Bolzano-Weierstrass Theorem.

2 2

Bounded, infinite subsets of $\mathbb{R}$ have at least one accumulation pt.

1

c. State the Monotone Convergence Theorem.

1 1

A monotone sequence converges if and only if it is bounded.

2 1 1

d. State the Ratio Test, proven in this class, concerning convergence of series. (Omit the third of the three statements that describes when the ratio test is inconclusive.) → not sequences, not power series

1 Suppose that the series $\sum a_n$ has nonzero terms.

1) if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ conv. abs

2) if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum a_n$ div.

1 1 1 1

2. (20 points) (5 points each) Definitions. Complete each sentence.

a. A sequence (a_n) *converges* to a if ...

$$\underbrace{\forall \varepsilon > 0}_{1}, \underbrace{\exists N}_{1} \text{ s.t. } \underbrace{n > N}_{1} \Rightarrow \underbrace{|a_n - a|}_{1} \underbrace{< \varepsilon}_{1}$$

b. A series $\sum_{n=1}^{\infty} a_n$ *converges* to s if ...

the sequence $S_n = a_1 + \cdots + a_n$ of $\underbrace{\text{partial sums}}_{3}$ $\underbrace{\text{converges to } s}_{2}$ as above.

$D

c. Given a function $f : D \to \mathbb{R}$, and an accumulation point c of the domain D, we say that

$$\lim_{x \to c} f(x) = L$$

if ...

$$\underbrace{\forall \varepsilon > 0}_{1}, \underbrace{\exists \delta > 0}_{1} \text{ s.t.}$$

$$\underbrace{x \in D \text{ and } 0 <}_{1} \underbrace{|x - c| < \delta}_{1} \Rightarrow \underbrace{|f(x) - L| < \varepsilon}_{1}.$$

d. The real number s is the *limit superior* of a bounded sequence (a_n) (that is, $s = \limsup a_n$) if ...

it is $\underbrace{\text{the supremum}}_{2}$ of the set of $\underbrace{\text{subsequential limits of } (a_n)}_{3}$.

3. (20 points) (5 points each) Calculations. No justification necessary.

a. Find the closure of the set

$$A = \bigcap_{n=1}^{\infty} (n, n+1).$$

$(A = \emptyset)$.

$\emptyset$

b. Find the set of subsequential limits of the sequence $(\frac{1}{1}, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \ldots)$.

$[0, 1]$

c. Find the sum of the convergent series

$$\sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^n.$$

(Note the index of the first term of the series.)

$$\sum_{n=0}^{\infty} \left(-\tfrac{1}{2}\right)^n = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3} \longleftarrow$$

3 pts for this or anything using $\frac{1}{1-r}$.

$$\sum_{n=2}^{\infty} \left(-\tfrac{1}{2}\right)^n = \frac{2}{3} - \left(1 - \tfrac{1}{2}\right) = \boxed{\frac{1}{6}}$$

d. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^n.$$

$$R = \lim \frac{(2n)!}{(n!)^2} \cdot \frac{((n+1)!)^2}{(2(n+1))!} = \lim \frac{(n+1)^2}{(2n+2)(2n+1)} = \boxed{\frac{1}{4}}$$

4. (20 points) (5 points each) Examples. No justification necessary. ALL OR NOTHING.

a. Give an example of a series that is convergent but not absolutely convergent.

alternating harmonic series.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

b. Give an example of a divergent p-series. That is, choose a p that makes the corresponding p-series divergent, and write the series in sum notation.

$$\sum_{n=1}^{\infty}\frac{1}{n^{1}} \leftarrow \text{or any other } p \leq 1.$$

c. Give an example of a sequence of irrational numbers that converges to a rational number.

$$a_n = \frac{\sqrt{2}}{n}. \quad (\to 0)$$

d. Give an example of an infinite collection $A_1, A_2, \ldots$ of open subsets of $\mathbb{R}$ with the property that

$$\bigcap_{n=1}^{\infty} A_n$$

is not open.

$$A_n = \left(-\frac{1}{n}, \frac{1}{n}\right), \quad n \geq 1.$$

$$\bigcap A_n = \{0\}, \text{ not open.}$$

5. (20 points) (20 points) Let A and B be sets, and let $f : A \to B$ be a function. Suppose that B_1 and B_2 are subsets of B. Prove that

$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2).$$

Pf ($\subseteq$) $a \in f^{-1}(B_1 \cup B_2)$

$\Longrightarrow f(a) \in B_1 \cup B_2$.

If $f(a) \in B_1$ then $a \in f^{-1}(B_1)$ and hence $a \in f^{-1}(B_1) \cup f^{-1}(B_2)$

Else $f(a) \in B_2$. Then $a \in f^{-1}(B_2)$ and hence $a \in f^{-1}(B_1) \cup f^{-1}(B_2)$

($\supseteq$) Suppose $a \in f^{-1}(B_1) \cup f^{-1}(B_2)$

If $a \in f^{-1}(B_1)$, then $f(a) \in B_1$, and hence $f(a) \in B_1 \cup B_2$, so $a \in f^{-1}(B_1 \cup B_2)$.

Else $a \in f^{-1}(B_2)$. Then $f(a) \in B_2$, and hence $f(a) \in B_1 \cup B_2$, so $a \in f^{-1}(B_1 \cup B_2)$.

5 structure of proving set equality

5 correct understanding of preimage.

5 correct understanding of union.

5 discretionary, form/logic.

6. (20 points) a. (5 points) Is $\mathbb{Q} \cap (0, 1)$ a countable set? Justify your answer directly from the definition of *countable*.

2

Yes. We exhibit a bijection $f : \mathbb{N} \to \mathbb{Q} \cap (0,1)$
(that is, a sequence that is 1-1 & onto)

1 $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \ldots$

informal description okay

2 lowest terms listed only $\Rightarrow$ 1-1.

b. (15 points) Is $\mathbb{Q} \cap (0, 1)$ a compact set? Justify your answer directly from the definition of *compact*.

5 No: here is an infinite cover of $\mathbb{Q} \cap (0,1)$ with no finite subcover:

3 $\{A_n : n \geq 2\}$ where $A_n = (\frac{1}{n}, 1 - \frac{1}{n})$

For any finite subcollection, let N be the largest index of all the included sets.

3 Then $\frac{1}{N} \in \mathbb{Q} \cap (0,1)$ but not in the union of the finite subcollection.
(OR, $\exists q \in \mathbb{Q} \cap (0, \frac{1}{N})$, since $\mathbb{Q}$ is dense in $\mathbb{R}$.)

7. (20 points) a. (15 points) Prove that, for all $n \geq 2$, we have

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

By induction:

3 when $n=2$, we have $\left(1-\frac{1}{2^2}\right)=\frac{3}{4}=\frac{2+1}{2\cdot 2}$.

3 Suppose the statement is true for $n=k$, and show it is true for $n=k+1$.

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)=\overbrace{\frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right)}^{3}$$

$$=\frac{k+1}{2k}\cdot\frac{k^2+2k+1-1}{(k+1)^2}=\frac{(k+1)\cdot k(k+2)}{2k(k+1)^2}$$

3

$$=\frac{k+2}{2(k+1)}=\underbrace{\frac{(k+1)+1}{2(k+1)}}_{3}.$$

b. (5 points) Let us say that the *infinite product*

$$\prod_{i=k}^{\infty} a_i$$

has value P if the sequence (s_n) of partial products

$$s_n=\prod_{i=k}^{n} a_i = a_k \cdot a_{k+1} \cdot \cdots \cdot a_n$$

converges to P. Find the value of the infinite product

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right).$$

$$=\lim_{n\to\infty}\frac{n+1}{2n}=\boxed{\frac{1}{2}}$$ all or nothing.

8. (20 points) (10 points each) Determine whether the following series converge. Show your work. If the series converges, find its sum.

a.

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

5 $\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$ (partial fractions)

$S_n = \frac{1}{2} - \frac{1}{3}$

5 $+ \frac{1}{3} - \frac{1}{4}$

$+ \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{2} - \frac{1}{n+2} \to \frac{1}{2}$.

converges to $\frac{1}{2}$.

b.

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)}$$

2 diverges

2 eventually $2n^2 > n^2 + 3n + 2$, and hence

2 eventually $\frac{n}{2n^2} < \frac{n}{n^2+3n+2}$

2 $\sum \frac{1}{2n}$ diverges (for if it converged, then so would $\sum \frac{1}{n}$)

2 By comparison test, $\sum \frac{n}{(n+1)(n+2)}$ diverges.

9. (20 points) Find the set of all real numbers x for which the following series converges:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}.$$

Show your work.

$$a_n = \frac{1}{n}$$

5 $R = \lim \frac{a_n}{a_{n+1}} = \lim \frac{n+1}{n} = 1.$

5 when $x = 3$, the series is $\sum \frac{1}{n}$, which diverges.

5 when $x = 1$, the series is $\sum \frac{(-1)^n}{n}$, which converges.

5 The series converges when $x \in [1, 3)$.

10. (20 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions that are continuous at $a \in \mathbb{R}$. Prove that the function $f + g$ is continuous at a.

(directly, not using sequences)

2 Let $\varepsilon > 0$ be given.

2 f cts at a: $\exists \delta_1 > 0$ s.t. if $|x-a| < \delta_1$, then $|f(x) - f(a)| < \frac{\varepsilon}{2}$.

2 g cts at a: $\exists \delta_2 > 0$ s.t. if $|x-a| < \delta_2$, then $|g(x) - g(a)| < \frac{\varepsilon}{2}$.

4 Let $\delta = \min\{\delta_1, \delta_2\}$.

2 If $|x-a| < \delta$, then

2 $|(f+g)(x) - (f+g)(a)|$

$= |f(x) - f(a) + g(x) - g(a)|$

2 $\leq |f(x) - f(a)| + |g(x) - g(a)|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

2 2

no ε-δ =