This exam contains 10 numbered problems. Check to see if any pages are missing. Point values are in parentheses. No books, notes, or electronic devices are allowed.

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1. (20 points) (4 points each) For each of the following five statements, determine whether the statement is true or false. Circle your answer. **No justification necessary.**

**ALL OR NOTHING**

(a) A conditional statement is logically equivalent to its contrapositive.

TRUE

(b) The set of rational numbers, together with the operations of addition and multiplication, is a complete ordered field.

TRUE

FALSE

not complete

(c) There exists a rearrangement of the terms of the alternating harmonic series that converges to zero.

TRUE

FALSE

(d) For every sequence \((a_n)\) of nonzero numbers, we have \(\limsup |a_n|^{1/n} \leq \limsup \frac{|a_{n+1}|}{|a_n|}.\)

TRUE

FALSE

(e) Every member of the set \(S = \{\frac{1}{n} : n \in \mathbb{N}\}\) is an isolated point of \(S.\)

TRUE

FALSE
2. (20 points) (5 points each) Statements.
   a. State the Principle of Mathematical Induction. (In your answer, let \( P(n) \) denote a statement about the natural number \( n \), and write your statement in the form "If \( \ldots \) and \( \ldots \), then \( \ldots \).")

   \[
   \begin{align*}
   &\text{If } P(1) \text{ is true,} & \quad & \text{---(1)} \\
   \quad & \text{and } P(k) \Rightarrow P(k+1) \text{ for all } k, & \quad & \text{---(2)} \\
   \quad & \text{then } P(n) \text{ is true for all } n. & \quad & \text{---(2)}
   \end{align*}
   \]

   b. State the Intermediate Value Theorem.

   \[
   \text{If } f: [a, b] \to \mathbb{R} \text{ is continuous,} \quad \text{---(1)}
   \]

   \[
   \text{and if } k \text{ is any number between } f(a) \text{ and } f(b), \quad \text{---(1)}
   \]

   \[
   \text{then there exists } c \text{ such that } a < c < b \text{ and } f(c) = k. \quad \text{---(2)}
   \]

   c. State the Heine-Borel Theorem.

   \[
   S \subseteq \mathbb{R} \text{ is compact if and only if } S \text{ is closed} \quad \text{---(1)}
   \]

   \[
   \Rightarrow \quad \text{and bounded.} \quad \text{---(2)}
   \]

   d. State the Completeness Axiom.

   \[
   \text{If } S \subseteq \mathbb{R} \text{ is nonempty and bounded above,} \quad \text{---(1)}
   \]

   \[
   \text{then } S \text{ has a supremum.} \quad \text{---(3)}
   \]
3. (20 points) (5 points each) Definitions. Complete each sentence.

a. A sequence \((a_n)\) converges to \(a\) if …

\[
\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. if } n \geq N \text{ then } |a_n - a| < \varepsilon.
\]

b. A series \(\sum_{n=1}^{\infty} a_n\) converges to \(s\) if …

the sequence \(s_n = a_1 + \ldots + a_n\) of partial sums converges to \(s\), as above.

c. A number \(x\) is a boundary point of a set \(S \subseteq \mathbb{R}\) if …

\[1\] \(\forall \varepsilon > 0,\)

\[2\] \(\mathbb{N}(x, \varepsilon) \cap S \neq \emptyset\) and

\[2\] \(\mathbb{N}(x, \varepsilon) \cap S^c \neq \emptyset.\)

d. Suppose that \(S \subseteq \mathbb{R}\) is nonempty and bounded below. The real number \(m\) is the infimum of the set \(S\) if …

\[2\] \(m \leq S, \forall s \in S.\)

and \(2\) if \(m' \leq S, \forall s \in S,\) then \(m' \leq m.\) (or contrapositive)
4. (20 points) (5 points each) Calculations. No justification necessary.

a. Find the limit of the convergent, recursively-defined sequence given by $a_1 = 5$ and $a_{n+1} = \sqrt{8a_n - 3}$.

\[ \text{the limit } a \text{ satisfies } a = \sqrt{8a - 3}. \]

\[ \Rightarrow a = 4 + \sqrt{13} \]

\[ \boxed{2} \text{ for solutions} \]

\[ \boxed{1} \text{ for identifying this one.} \]

(4 - $\sqrt{13}$ is an extraneous root, because the sequence increases from 5)

b. Find the set $S$ of subsequential limits and find $\limsup a_n$ and $\liminf a_n$ for the sequence

\[ (a_n) = \left( \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \ldots \right). \]

\[ \Rightarrow S = \{ 0, 2 \} \]

\[ \liminf a_n = \limsup a_n = 0. \]

\[ \boxed{1} \text{ based on } S. \]

c. Find the sum of the convergent series

\[ \sum_{n=3}^{\infty} \left( \frac{1}{3} \right)^n. \]

(Note the index of the first term of the series. Write your answer in the form $\frac{m}{n}$, where $m$ and $n$ are natural numbers.)

\[ \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^n = 1 - \frac{1}{3} = \frac{2}{3} \]

\[ \boxed{3} \]

\[ \sum_{n=3}^{\infty} \left( \frac{1}{3} \right)^n = \frac{8}{2} - 1 - \frac{1}{3} - \frac{1}{9} = \frac{5}{18} \]

\[ \boxed{2} \]

d. Write the boundary and interior of the set of rational numbers, as a subset of the real numbers.

\[ \partial \mathbb{Q} = \mathbb{R} \]

\[ \text{int } \mathbb{Q} = \emptyset \]

\[ \boxed{3} \]

\[ \boxed{2} \]
5. (20 points) (5 points each) Examples. No justification necessary. **ALL OR NOTHING**

a. Give an example of a series $\sum_{n=1}^{\infty} a_n$ that is convergent and has terms $a_n$ that are constant.

$$a_n \to 0 \text{ is a necessary condition for convergence of } \sum_{n=1}^{\infty} a_n.$$  

The only sequence that is constant & converges to 0 is $a_n = 0$.  

$$\sum_{n=1}^{\infty} 0$$

b. Give an example of a sequence $(a_n)$ whose terms are all distinct and positive, and $\limsup a_n = +\infty$ and $\liminf a_n = 0$.

$$(a_n) = (2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots)$$

c. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ and a subset $S \subseteq \mathbb{R}$ with the property that $f^{-1}(f(S)) \neq S$.

$$f(x) = x^2$$  

$S = \{1, 2\}$  

$$f^{-1}(f(S)) = \{-1, 1\} \neq S.$$

d. Give an example of an infinite collection $C_1, C_2, \ldots$ of closed subsets of $\mathbb{R}$ with the property that

$$\bigcup_{n=1}^{\infty} C_n$$

is not closed.

$$C_n = \left[\frac{1}{n}, 3 - \frac{1}{n}\right] \text{ closed}$$

$$\bigcup_{n=1}^{\infty} C_n = (0, 3) \text{ not closed}.$$
6. (20 points) Let $A$ and $B$ be sets, and let $f : A \to B$ be a function. Suppose that $A_1$ and $A_2$ are subsets of $A$. Prove that
\[ f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2). \]

\[ \text{Pf} \quad \text{Let } y \in f(A_1 \cap A_2). \]

There exists $x \in A_1 \cap A_2$ with $f(x) = y$.  

$x \in A_1 \Rightarrow y \in f(A_1)$.  

$x \in A_2 \Rightarrow y \in f(A_2)$.  

$\Rightarrow y \in f(A_1) \cap f(A_2)$.  

$\blacksquare$
7. (20 points) a. (15 points) Prove that for all \( n \), we have

\[
(2)(6)(10)(14) \cdots (4n - 2) = \frac{(2n)!}{n!}.
\]

When \( n = 1 \), we have \( 2 = \frac{2!}{1!} \).\(\text{3}\)

Suppose that \( (2)(6) \cdots (4k-2) = \frac{(2k)!}{k!} \), and show the statement is true for \( n = k+1 \).

\[
(2)(6) \cdots (4(k+1) - 2) = (2)(6) \cdots (4k-2)(4k+2)
\]

\[
= \frac{(2k)!}{k!} (4k+2) \quad \text{by hypothesis}.\(\text{3}\)
\]

\[
= \frac{(2k)!}{k!} \cdot 2 \cdot (2k+1) = \frac{(2k+1)!}{k!} \cdot 2\(\text{3}\)
\]

\[
= \frac{(2k+1)!}{k!} \cdot \frac{2(k+1)}{k+1} = \frac{(2k+2)!}{(k+1)!}\(\text{algebra}\)
\]

\[
= \frac{(2(k+1))!}{(k+1)!} \quad \text{\(\text{2}\) form}
\]

Thus, the statement is true for all \( n \), by the principle of mathematical induction.

b. (5 points) Let us say that the infinite product

\[
\prod_{i=1}^{\infty} a_i
\]

has value \( P \) if the sequence \( (s_n) \) of partial products

\[
s_n = \prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdots \cdot a_n
\]

converges to \( P \). Find the value of the infinite product

\[
\prod_{n=1}^{\infty} \frac{1}{4n-2}.
\]

By (a), we have

\[
S_n = \frac{n!}{(2n)!}.
\]

By (a), we have

\[
\frac{S_{n+1}}{S_n} = \frac{n+1}{(2n+1)(2n+2)} \to 0.
\]

By ratio test, \( S_n \to 0 \).

Thus \( \prod \frac{1}{4n-2} = 0 \).\(\text{2}\)
8. (20 points) a. (5 points) Complete the following definition: \( f : \mathbb{R} \to \mathbb{R} \) is continuous at \( x = a \) if \( \ldots \) (Note: this definition contains nested quantifiers. Also note that \( \mathbb{R} \) is the domain of \( f \).)

\[
\forall \varepsilon > 0, \ \exists \delta > 0 \text{ such that } \frac{\lvert x - a \rvert < \delta}{\text{if } \lvert f(x) - f(a) \rvert < \varepsilon}. \]

b. (5 points) Write the negation of your definition in (a).

\[
\exists \varepsilon > 0 \text{ such that } \forall \delta > 0, \ \exists x \text{ such that } \frac{\lvert x - a \rvert < \delta}{\text{AND } \lvert f(x) - f(a) \rvert \geq \varepsilon}. \]

c. (10 points) Show, directly from the definition of continuity, that the function \( f(x) = x^2 - 3x \) is continuous at \( x = 2 \).

Note that on \((1, 3)\), we have \( \lvert x - 1 \rvert < 2 \).

Let \( \varepsilon > 0 \) be given. Choose \( \delta = \min \left\{ 2, \frac{\varepsilon}{2} \right\} \).

Then if \( \lvert x - 2 \rvert < \delta \),

\[
\lvert f(x) - f(2) \rvert = \lvert x^2 - 3x - (-2) \rvert = \lvert (x - 2)(x - 1) \rvert = \lvert x - 2 \rvert \cdot \lvert x - 1 \rvert \leq 2 \lvert x - 2 \rvert, \text{ since } \delta \leq 1.
\]

\[
< 2 \delta \leq 2 \cdot \frac{\varepsilon}{2} = \varepsilon.
\]
9. (20 points) Find the set of all real numbers \( x \) for which the following series converges:

\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{x+5}{3} \right)^n.
\]

Show your work.

If \( b_n = \frac{1}{n} \left( \frac{x+5}{3} \right)^n \),

then \( \left| \frac{b_{n+1}}{b_n} \right| = \frac{n}{n+1} \cdot \frac{1}{3} \left| \frac{x+5}{3} \right| \)

\( \to \left| \frac{x+5}{3} \right| < 1 \) ?

If \( \left| \frac{x+5}{3} \right| < 1 \),

then the series is a harmonic series that diverges.

When \( x = -2 \), \( -8 < x < 2 \)

It is also an alternating harmonic series that converges.

The power series converges when \( x \in [-8, 2) \).

Alternately:

\( \left| b_n \right|^{\frac{1}{n}} = \frac{1}{3} \left| \frac{x+5}{3} \right| \to \frac{1}{3} \left| \frac{x+5}{3} \right| \) & apply root test.
10. (20 points) (10 points each) Determine whether each of the following series converges absolutely, converges conditionally, or diverges. Justify your answers. In the case of convergence, do not find the sum.

a. 
\[
\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^4 + 2}}
\]

\[
|a_n| = \frac{n}{\sqrt{n^4 + 2}} > \frac{\sqrt{n^4 + 2}}{\sqrt{n^4 + 2}} = \frac{1}{n \cdot \sqrt{3}}
\]

so \(\sum |a_n|\) diverges by the comparison test.

However, since \(\frac{n}{\sqrt{n^4 + 2}} \to 0\) and is a decreasing function of \(n\) (no justification needed), \(\sum a_n\) converges by the alternating series test. Thus \(\sum a_n\) converges conditionally.

b. 
\[
\frac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{2^2 \cdot 3^2} - \frac{1}{2^2 \cdot 3^2} + \frac{1}{2^3 \cdot 3^2} - \cdots
\]

\[
\left| \frac{a_{n+1}}{a_n} \right| = \begin{cases} 
\frac{1}{3}, & \text{if } n \text{ is odd} \\
\frac{1}{2}, & \text{if } n \text{ is even}
\end{cases}
\]

\[
\lim \sup \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2} < 1 \quad \text{and hence}
\]

\(\sum a_n\) converges absolutely by the ratio test.