This exam contains 7 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you are applying a theorem, you should indicate this fact**, and explain why the theorem may be applied.
- **Do not trivialize a problem.** If you are asked to prove a theorem, you cannot just cite that theorem.
- **Organize your work** in a reasonable, tidy, and coherent way. Work that is disorganized and jumbled that lacks clear reasoning will receive little or no credit.
- **Unsupported answers will not receive full credit.** An answer must be supported by calculations, explanation, and/or algebraic work to receive full credit. Partial credit may be given to well-argued incorrect answers as well.
- If you need more space, use the back of the pages. Clearly indicate when you have done this.

Do not write in the table to the right.

### Solutions

You may use the following matrices and computations on the exam without defining or proving them.

\[
R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad F_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}, \quad R_\phi R_\theta = R_{\phi+\theta}, \quad F_\phi F_\theta = R_{2(\phi-\theta)}
\]

\[
\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta), \quad \sin(2\theta) = 2\sin(\theta)\cos(\theta)
\]

Suppose \( \ell \) forms an angle of \( \theta \) with the horizontal. If \( U \parallel \ell \), then \( F_\theta U = U \). If \( V \perp \ell \), then \( F_\theta V = -V \).
1. Let $U = (2, 3)$ and $V = (1, 4)$.
   
   (a) (1 point) Compute $U \cdot V$.
   
   $U \cdot V = 2 \cdot 1 + 3 \cdot 4 = 2 + 12 = 14$
   
   (b) (1 point) Compute $U + V$.
   
   $U + V = (2+1, 3+4) = (3, 7)$
   
   (c) (1 point) Compute $\|U\|$.
   
   $\|u\| = \sqrt{U \cdot U} = \sqrt{4+9} = \sqrt{13}$

2. Let $\ell$ be the line in the picture below.
   
   (a) (4 points) Find a parametric equation for $\ell$.
   
   (Other answers possible)
   
   $P = (0,-2) \in \ell$
   
   $Q - P = (3,0) - (0,-2) = (3,2)$
   
   is a Dirin Indicator of $\ell$
   
   $P + t(Q-P) = (0,-2) + t(3,2)$

   (b) (4 points) Find a normal equation for $\ell$.
   
   (Other answers possible)
   
   $(3,2) \parallel \ell$, so $A = (-2,3) \perp \ell$
   
   $(-2,3) \cdot (X - (0,-2)) = 0$. 
3. Recall that \( \arccos z = \int_z^1 \frac{1}{\sqrt{1-t^2}} dt \) and we define \( \pi = \int_{-1}^1 \frac{1}{\sqrt{1-t^2}} dt \).

(a) (6 points) Let rays \( p \) and \( q \) emanate from a common vertex, with direction indicators \((-3,4)\) and \((2,3)\). Find \(|\angle(p,q)|\). An answer with an integral is fine.

\[
\text{We need unit dirn vectors : } U = \frac{(-3,4)}{||(-3,4)||} = \frac{(-3,4)}{5} \quad \text{ and } \quad V = \frac{(2,3)}{||(2,3)||} = \frac{(2,3)}{\sqrt{13}}.
\]

\[
\frac{1}{||U||^2} \int_{U} \frac{1}{\sqrt{1-t^2}} dt = \int_{V/\sqrt{13}} \frac{1}{\sqrt{1-t^2}} dt.
\]

\[
\angle(p,q) = \sqrt{\frac{1}{5}} \int_0^1 \frac{1}{\sqrt{1-t^2}} dt = \frac{1}{2} \int_{-1}^1 \frac{1}{\sqrt{1-t^2}} dt = \frac{1}{2} \pi.
\]

\[
\text{even function}
\]

(b) (6 points) Use calculus to prove that \( \arccos(0) = \pi/2 \).

\[
\arccos(0) = \int_0^1 \frac{1}{\sqrt{1-t^2}} dt = \frac{1}{2} \int_{-1}^1 \frac{1}{\sqrt{1-t^2}} dt = \frac{1}{2} \pi.
\]

(c) (6 points) In the picture below, \( U = (1,0) \), \( V = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \), \( W = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \), and \( X = (-1,0) \). Prove \(|\angle UOV| = \pi/3 \) using the methods of this course.

\[
\alpha = \arccos(U \cdot V) = \arccos\left(\frac{1}{2}\right) \quad \beta = \arccos(V \cdot W) = \arccos\left(-\frac{1}{2}\right) \quad \gamma = \arccos(W \cdot X) = \arccos\left(-\frac{1}{2}\right)
\]

Thus \( \alpha = \beta = \gamma \). Since \( \alpha + \beta + \gamma = \pi \), we have

\[
\alpha + \beta + \gamma = 3\alpha = \pi \Rightarrow \alpha = \pi/3
\]
4. (8 points) In the diagram below, \( j \parallel k \). Prove \( |\angle PQR| = |\angle SRT| \) using the definitions and methods of this course.

Let \( U = \frac{T-R}{||T-R||} \) and \( V = \frac{S-R}{||S-R||} \). Then \( U \parallel l \), hence is a dirn indicator of \( \angle TRS \) and of \( \angle PQR \).

Similarly, \( V \parallel j \) (hence also \( k \)) and is the other DI for each angle. Thus

\[
|\angle PQR| = \arccos(U \cdot V) = |\angle SRT|
\]

5. Prove the following facts about vectors using methods from this class.

   (a) (5 points) Given two vectors \( U \) and \( V \), prove \( ||U + V||^2 = ||U||^2 + ||V||^2 + 2U \cdot V \).

\[
||U + V||^2 = (U + V) \cdot (U + V) = U \cdot U + V \cdot V + U \cdot V + V \cdot U \\
= ||U||^2 + ||V||^2 + 2U \cdot V
\]

   (b) (5 points) If \( U \) and \( V \) are unit vectors and \( U \cdot V = -1 \), then \( U = -V \).

\[
||U|| = ||V|| = 1. \text{ If } U \cdot V = -1, \text{ then by part (a):}
\]

\[
||U + V||^2 = 1 + 1 - 2 = 0.
\]

Thus \( ||U + V|| = 0 \Rightarrow U + V = 0 \Rightarrow U = -V \).
6. (a) (3 points) Complete the definition: \( U(X) \) is an isometry of \( \mathbb{R}^2 \) if...

\[
\| U(p) - U(q) \| \quad \forall \ p, q \in \mathbb{R}^2
\]

(b) (4 points) Let \( T(X) = X + V \) be translation by the vector \( V \). Prove \( T \) is an isometry.

\[
\forall p, q \in \mathbb{R}^2 : \quad \| T(p) - T(q) \| = \| p + V - (q + V) \|
\]

\[
= \| p + V - q - V \|
\]

\[
= \| p - q \|
\]

(c) (6 points) Let \( U \) and \( V \) be isometries of \( \mathbb{R}^2 \). Prove that the composition \( U \circ V \) is an isometry.

\[
\forall p, q \in \mathbb{R}^2 : \quad \| U(V(p)) - U(V(q)) \| = \| V(p) - V(q) \|
\]

\[
= \| p - q \|
\]

(d) (12 points) Let \( \ell = \{(1, 3) + t(4, 2)\} \), where \( t \in \mathbb{R} \). Find the matrix formula \( M_\ell(X) \) for the reflection across the line \( \ell \). Your answer should include exact values in the matrix, not trig functions, but you do not need to multiply out your entire formula.

\[
M_\ell(x) = F_\theta(x-P) + P = \begin{bmatrix} \frac{3}{\sqrt{5}} & \frac{4}{\sqrt{5}} \\ \frac{4}{\sqrt{5}} & -\frac{3}{\sqrt{5}} \end{bmatrix}(x - \begin{bmatrix} 1 \\ 3 \end{bmatrix}) + \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\]

\[
\cos \theta = \frac{4}{\sqrt{20}}
\]

\[
\sin \theta = \frac{2}{\sqrt{20}}
\]

\[
\cos 2\theta = \left(\frac{4}{\sqrt{20}}\right)^2 - \left(\frac{2}{\sqrt{20}}\right)^2 = \frac{16 - 4}{20} = \frac{12}{20} = \frac{3}{5}
\]

\[
\sin 2\theta = 2\left(\frac{4}{\sqrt{20}}\right)\left(\frac{2}{\sqrt{20}}\right) = \frac{16}{20} = \frac{4}{5}
\]
7. In this problem we'll consider the composition of reflections, e.g. \( M_m \circ M_\ell(X) = M_m(M_\ell(X)) \). If you use any angles which are not in the diagrams, make sure to define and label them.

(a) (8 points) In the diagram below, lines \( \ell \) and \( m \) intersect at \( C \), with angles \( \alpha \) and \( \beta \) as shown below. The dotted line is horizontal, i.e. parallel to the \( x \)-axis. Use the methods of this course to prove \( M_m \circ M_\ell(X) \) is a rotation about \( C \) by an angle of \( 2\beta \).

\[
M_m(M_\ell(X)) = M_\ell(F_\alpha(x-C)+C) \\
= F_{\alpha+\beta}(F_\alpha(x-C)+C) - C + C \\
= R_{2\alpha}(x-C) + C
\]

(By cover page, \( F_{\alpha+\beta}F_\alpha = F_\alpha(\alpha+\beta) = F_{2\alpha} \))

(b) (8 points) In the diagram below, \( j \parallel k \), and points \( P \in j \) and \( Q \in k \) are chosen so that the vector \( Q-P \) is perpendicular to both \( j \) and \( l \). The dotted line is horizontal, i.e. parallel to the \( x \)-axis. Use the methods of this course to prove \( M_k \circ M_j \) is a translation by \( V = 2(Q-P) \).

\[
M_k(M_j(X)) = M_\ell(F_\theta(x-P)+P) \\
= F_\theta(F_\theta(x-P)+P) - Q + Q \\
= F_\theta(F_\theta(x-P)) + F_\theta(P-Q) + Q \\
= I - (P-Q) \\
= X - P - P + Q + Q \\
= X + 2Q - 2P \\
= X + 2(Q-P)
\]
8. (12 points) Indicate whether each statement is True or False by circling the appropriate answer. Justify your answer with definitions, theorems and methods from this course. (If false, be specific with your explanation; e.g. tell me what part of a definition or theorem is not satisfied, or give an example to show the statement is false, etc.)

(a) For any \( A, B \) and \( C \), the barycentric coordinates of the origin \((0,0)\) are always \((0,0,0)_{\triangle ABC}\).

\[(0,0)_{\triangle ABC} \text{ not valid BC's; } 0+0+0 = 1\]

(b) If \( \triangle ABC \cong \triangle A'B'C' \), there exist two different isometries which send \( A \) to \( A' \), \( B \) to \( B' \), and \( C \) to \( C' \).

\( \exists \) unique such isometry, by Chapter 4.
(Thms 4.17, 4.18)

(c) Any isometry can be constructed as the composition of two or fewer reflections.

Three might be required. (Lab 3, for example— or any glide reflection)