The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work. Please let me know if you spot any typos and I'll update things as soon as possible.

I've tried to indicate where each problem came from in an effort to help you learn how I write exams, since the second midterm is about one month away.

(1) On any test I write, Problem 1 is generally intended to be fairly straightforward, where everybody will get almost all of the points and be settled down for the rest of the exam. Unfortunately most answered a different question, about congruence instead of equality. The = in the problem really was intended to be there. I wanted to know when the set of points designated by $\overline{PQ}$ is equal to the set of points $\overline{RS}$, not when they're congruent.

Assume $P \neq Q$ or the symbols here won't even make sense. The key to this problem is that the order of endpoints in a line segment doesn't matter, but it does with a ray because a ray has a starting point. Also, the ray from P to Q may go through S, even if $Q \neq S$.

(a) ... ($P = R$ and $Q = S$) or ($P = S$ and $Q = R$). (More succinctly, $\{P, Q\} = \{R, S\}$.)

(b) ... $P = R$ and $S - R$ is a positive multiple of $Q - P$.

Note that this is problem #1.36, i.e. problem 36 from the Chapter 1 exercises. It wasn't assigned as homework, but I'll occasionally mine the exercises for extra test problems because they're similar to what you've already read and worked on.

(2) This is (half of) Lemma 26, whose proof is requested in problem #1.15, but I did it in class. It is simpler if you don't write out the vector in terms of its components.

$$\begin{aligned} ||X+Y||^2 &= \langle X+Y, X+Y\rangle \\ &= \langle X, X\rangle + \langle Y, Y\rangle + \langle X, Y\rangle + \langle Y, X\rangle \\ &= ||X||^2 + ||Y||^2 + 2\langle X, Y\rangle \end{aligned}$$

(3) I tried to warn everybody as often as possible that this type of problem would show up, and most people did extremely well with it. Answers:

$$\begin{aligned} P_1 &= (+,+,+)^{\triangle ABC} \\ P_2 &= (0,+,+)^{\triangle ABC} \\ P_3 &= (+,-,+)^{\triangle ABC} \\ P_4 &= (-,+,-)^{\triangle ABC} \end{aligned}$$

(4) (a) We covered angle measure fairly extensively in class. It's also very similar to #2.7, which I suggested as a review problem. First we need unit vectors which describe the angle:

$$\begin{aligned} U &= (A-B)/||A-B|| = (2,0)/2 = (1,0) \\ V &= (C-B)/||C-B|| = (1,2)/\sqrt{5} = (1/\sqrt{5}, 2/\sqrt{5}) \end{aligned}$$

Then the measure of the angle is

$$\arccos\left(\langle U, V\rangle\right) = \arccos\left(1/\sqrt{5}\right) = \int_{1/\sqrt{5}}^{1} \frac{1}{\sqrt{1-t^2}} dt$$

The most common mistakes were using A and B instead of $A-B$ and $A-C$, or forgetting to normalize them.

(b) This is a modified version of problem #3.28 from your homework. Here are two possible approaches, either of which involve a rotation and a translation.

Method 1: Rotate about the origin by 90°; then move up one unit:

$$\mathcal{U}(X) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Method 2: Move left one unit, rotate about the origin by 90°; then move up two units:

$$\mathcal{U}(X) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left(X - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

If you carry out the matrix multiplications and add the terms together, you'll find that both formulas give the same isometry $\mathcal{U}(x, y) = (-y, x + 1)$.

(5) (a) There are many ways to do this problem, involving either the distance formula or some sort of proportionality. The simplest for me is

$$\begin{aligned} R &= P + (3/4)(Q - P) \\ &= (1, 0) + (3/4)(2, 4) \\ &= (1, 0) + (3/2, 3) = (5/2, 3) \end{aligned}$$

I suggested problems #1.38 and #1.39 as review exercises, both of which are similar to this but harder. Most people did well, occasionally forgetting to add $(1, 0)$ to $(3/4)(Q - P)$.

(b) This problem was unintentionally tricky. Remember that the *normal* form of a line k through R is $\{\langle A, X - R\rangle = 0\}$ where $A \perp k$. Most people used a *direction indicator* of k for A instead of a normal vector.

$Q - P = (2, 4)$ is a direction indicator for the line l, so it is perpendicular to any line k which is perpendicular to l. Hence:

$$k = \{\langle Q - P, X - R\rangle = 0\} = \{\langle (2, 4), X - (5/2, 3)\rangle\}$$

(c) This problem was intended to be a little tricky, and certainly turned out that way. The intention was to see something like the proof of Proposition 11 in Chapter 1, which I did in class, although it would be simpler because you could use the specific lines here. In the end most people had a good feel for what was going on but wrote out a proof which essentially assumed Proposition 11 (or similar results) – essentially asserting what was to be proved. For consistency I gave most people one point, although a few earned two.

For the record: To show that k is the unique such line, we need to suppose that m is some other line through R perpendicular to l and show that $k = m$. Because k and m share the point R, this amounts to showing they have parallel direction vectors (or equivalently, if you prefer, parallel normal vectors). So let U and V be direction vectors for k and m, respectively. Because both lines are perpendicular to l, U and V are perpendicular to $Q - P = (2, 4)$:

$$\begin{aligned} \langle U, (2, 4)\rangle &= 2u_1 + 4u_2 = 0 \\ \langle V, (2, 4)\rangle &= 2v_1 + 4v_2 = 0 \end{aligned}$$

Rearranging those equations give:

$$u_1 = -2u_2$$
$$v_1 = -2v_2$$

In other words, both U and V have the form $(-2a, a)$ – i.e. the first component is -2 times the second component. Thus they must be multiples of each other.

(6) (a) An altitude is a line through a vertex which is perpendicular to the line containing the opposite side of the triangle. A median is a point connecting a vertex with the midpoint of the opposite side.

(b) This was a proof done in class. The fastest approach is to recall that the barycentric coordinates of the centroid are $(1/3, 1/3, 1/3)^{\triangle ABC}$ and to show that this point is on each median. I'm happy to work through this problem in more detail, but you can simply parametrize the medians using barycentric coordinates. Here's the median from A to the midpoint L of $\overline{BC}$.

$$m(t) = \left(t, \frac{1-t}{2}, \frac{1-t}{2}\right)^{\triangle ABC} \quad \text{where } t \in [0, 1]$$

Note that $m(1) = (1, 0, 0)^{\triangle ABC} = A$, and $m(0) = (0, 1/2, 1/2) = L$, so this line segment stretches between A and L as desired. Now let $t = 1/3$:

$$m(1/3) = \left(1/3, \frac{2/3}{2}, \frac{2/3}{2}\right)^{\triangle ABC} = (1/3, 1/3, 1/3)^{\triangle ABC}$$

So that point is in fact on the median. A symmetric argument works for the other two medians, so we're done.

(7) This is very similar to problem #2.16 on your homework. Here's one possible solution: in addition to $U = (0, 1)$ and $V = (-1/2, \sqrt{3}/2)$, define two more vectors:

$$W = (-\sqrt{3}/2, 1/2) \qquad Y = (-1, 0)$$

(Draw a picture of all four vectors!) You can check that

$$\langle U, V\rangle = \sqrt{3}/2$$
$$\langle V, W\rangle = \sqrt{3}/2$$
$$\langle W, Y\rangle = \sqrt{3}/2$$

This implies $|\angle UOV| = |\angle VOW| = |\angle WOY|$. Since these angles are all adjacent to each other, we also know that

$$|\angle UOV| + |\angle VOW| + |\angle WOY| = |\angle UOY|$$

But $\langle U, Y\rangle = 0$, so $\angle UOY$ is a right angle! Hence we have three equal angles whose measures add up to $\pi/2$, proving that $|\angle UOV| = \frac{\pi/2}{3} = \pi/6$.