The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

4.11.3: Since $||\triangle ABC|| = \frac{1}{4}\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}$, we see that

$$16||\triangle ABC||^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4.$$

Hence we can rewrite the formula for the orthocenter in Proposition 7 as

$$\left(\frac{a^4 - (b^2 - c^2)^2}{16||\triangle ABC||^2}, \frac{b^4 - (c^2 - a^2)^2}{16||\triangle ABC||^2}, \frac{c^4 - (a^2 - b^2)^2}{16||\triangle ABC||^2}\right)^{\triangle}$$

4.11.4: Heron's Formula says $||\triangle ABC|| = \sqrt{\frac{p}{2}\left(\frac{p}{2} - a\right)\left(\frac{p}{2} - b\right)\left(\frac{p}{2} - c\right)}$. We're given $a = 7$, $b = 13$, and $c = 17$, and can calculate that $p = a+b+c = 37$. Hence $||\triangle ABC|| = \sqrt{28083/16} \approx 41.895$ square units.

4.11.7: We have the following barycentric coordinates for the centroid G, the orthocenter H and the circumcenter J:

$G = (1/3, 1/3, 1/3)^{\triangle}$ (from Theorem 3)

$$H = \left(\frac{a^4 - (b^2 - c^2)^2}{16||\triangle ABC||^2}, \frac{b^4 - (c^2 - a^2)^2}{16||\triangle ABC||^2}, \frac{c^4 - (a^2 - b^2)^2}{16||\triangle ABC||^2}\right)^{\triangle} \quad \text{(from Problem 3)}$$

$$J = \left(\frac{a^2(b^2 + c^2 - a^2)}{16||\triangle ABC||^2}, \frac{b^2(c^2 + a^2 - b^2)}{16||\triangle ABC||^2}, \frac{c^2(a^2 + b^2 - c^2)}{16||\triangle ABC||^2}\right)^{\triangle} \quad \text{(from Proposition 15)}$$

Proposition 10 in Chapter 3 says we can compute $\frac{1}{3}H + \frac{2}{3}J$ "coordinate-wise." From here on out the problem becomes one of algebra: tedious but straightforward. For example, the work for the first coordinate is:

$$\begin{aligned}
\frac{1}{3}\left(\frac{a^4 - (b^2 - c^2)^2}{16||\triangle ABC||^2}\right) + \frac{2}{3}\left(\frac{a^2(b^2 + c^2 - a^2)}{16||\triangle ABC||^2}\right) &= \frac{a^4 - (b^2 - c^2)^2 + 2a^2(b^2 + c^2 - a^2)}{48||\triangle ABC||^2} \\
&= \frac{a^4 - b^4 + 2b^2c^2 - c^4 + 2a^2b^2 + 2a^2c^2 - 2a^4}{48||\triangle ABC||^2} \\
&= \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{48||\triangle ABC||^2} \\
&= \frac{16||\triangle ABC||^2}{48||\triangle ABC||^2} = 1/3
\end{aligned}$$

4.11.8: Suppose the orthocenter of $\triangle ABC$ is a vertex – say C. Then the altitudes from A and B must meet at C, which means $\overline{AC}$ and $\overline{BC}$ are perpendicular. Hence this happens if and only if $\triangle ABC$ has a right angle at C. (If you didn't get this problem correct, draw a picture. Also draw in the altitude from C and make sure you realize why it's concurrent with the other two altitudes.)

4.11.10: If $\triangle ABC$ is equilateral, then $a = b = c$, so the incenter (by Theorem 18) is

$$\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{a}{a+b+c}\right)^{\triangle} = \left(\frac{a}{a+a+a}, \frac{a}{a+a+a}, \frac{a}{a+a+a}\right)^{\triangle} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\triangle},$$

which is the centroid. Conversely, suppose the centroid and incenter are equal:

$$\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{a}{a+b+c}\right)^{\triangle} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\triangle}$$

Equating the parts of the barycentric coordinates (and multiplying by 3) gives three equations:

$$3a = a + b + c$$
$$3b = a + b + c$$
$$3c = a + b + c$$

You can check that this system of equations is satifsfied if and only if $a = b = c$, so that the triangle is equilateral. (Come talk to me if you had trouble with this system.)

4.11.19: If you label $A = (0, 0)$, $B = (5, 0)$ and $C = (-4, -4)$, you can calculate that

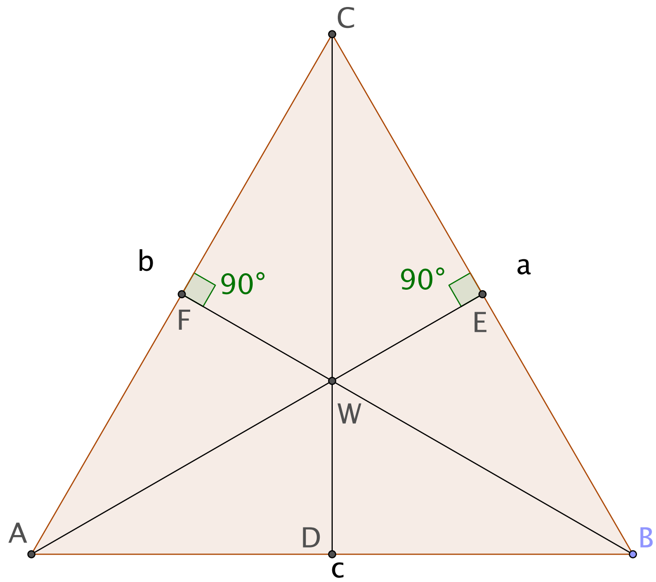
$$a = |\overline{BC}| = \sqrt{97}$$
$$b = |\overline{AC}| = 2\sqrt{2}$$
$$c = |\overline{AB}| = 5$$

After that you can use formulas throughout the chapter to calculated the requested points. To find the equation take any two points which you know to be on the Euler line, and find the equation of the line through them.

4.11.26: If $\triangle ABC$ is equilateral, then $a = b = c$, so you can use the barycentric coordinates of the incenter and orthocenter to show these are both (after much simplication) equal to $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\triangle}$.

In the other direction, we assume that the incenter equals the orthocenter and must show that the triangle is equilateral. One way is to set up a huge system of equations where the barycentric coordinates of the orthocenter are equal to the barycentric of the coordinates of the incenter, and show this is only true of $a = b = c$. That's possible but messy. A geometric proof seems simpler, except that it's hard without the ASA congruence theorem for triangles, which makes its appearance in Chapter 5. Here's a hybrid approach, which uses a bit of geometry followed by algebraic computations.

Suppose the incenter and orthocenter are the same point; in the following picture I'll refer to it as W. First, note that the angle bisectors must equal the altitudes, which is why I only have one line emanating from each vertex. (Why is this the case? Consider the vertex C on top. The angle bisector and altitude each start at C and go through W by our assumption. Because they share two points, they must be contained in the same line!)



Because $\overline{CW}$ is an angle bisector, $|\angle ECW| = |\angle FCW|$. Together with the knowledge that

$$|\angle WEC| = |\angle WFC| = \pi/2$$

(since $\overline{AE}$ and $\overline{BF}$ are altitudes), we can conclude that $|\angle EWC| = |\angle FWC|$, since the angles in each of $\triangle CWE$ and $\triangle CWF$ must sum to π.

Furthermore, by Proposition 17, W is equidistant from $\overline{CA}$ and $\overline{CB}$, which means $|\overline{WF}| = |\overline{WE}|$. Hence by the SAS congruence criterion (Theorem 27 in Chapter 3),

$$\triangle CWE \cong \triangle CWF$$

It immediately follows that $\overline{FC} \cong \overline{EC}$.

Here's the algebraic part of my proof. By Proposition 5, the barycentric coordinates of E and F are

$$E = \left(0, \frac{a^2+b^2-c^2}{2a^2}, \frac{a^2+c^2-b^2}{2a^2}\right)^{\triangle}$$
$$F = \left(\frac{b^2+a^2-c^2}{2b^2}, 0, \frac{b^2+c^2-a^2}{2a^2}\right)^{\triangle}$$

and, of course, $C = (0,0,1)^{\triangle}$. If you use the barycentric distance formula (Proposition 28 in Chapter 3) you can check

$$|\overline{EC}| = |\overline{FC}|$$
$$\frac{(a^2+b^2-c^2)^2}{4a^2} = \frac{(a^2+b^2-c^2)^2}{4b^2}$$

That equality could hold two ways: (1) the numerators are both 0, so $a^2+b^2=c^2$ and we have a right triangle. But this is impossible because the orthocenter equals the incenter and hence is in the interior (see problem 8)! The other possibility is (2) $a = b$. By symmetry, you could show that $b = c$ and $a = c$, finishing the proof.