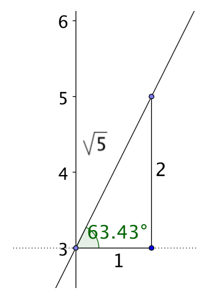
The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

7.9.6,7: The translation in these problems is easy to write a formula for: $\mathcal{T}(X) = X + (-3, -6)$ or, if you prefer, $\mathcal{T}(x, y) = (x-3, y-6)$. The equation for the reflection is trickier. We know $\langle(2, -1), (x, y)\rangle = -3$ is equivalent to $2x - y = -3$, or $y = 2x + 3$. So the slope of the mirror is 2, and hence the angle it forms with a horizontal line is $\theta = \arctan 2 \approx 63.4°$. The y-intercept form of the line makes it clear that $(0, 3)$ is on the line. Hence a matrix formula for the reflection across this line is:

$$\mathcal{M}(X) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \left([X] - \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Here's a picture of the line:



Using the triangle in the picture, we see that

$$\cos 2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5}$$

$$\sin 2\theta = 2\cos\theta\sin\theta = \frac{4}{5}$$

Hence our formula for the reflection becomes

$$\mathcal{M}(X) = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \left([X] - \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Now consider the two compositions (check the parentheses carefully to make sure you see the differences!):

$$\mathcal{T} \circ \mathcal{M}(X) = \left(\begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \left([X] - \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) + \begin{bmatrix} -3 \\ -6 \end{bmatrix}$$

$$\mathcal{M} \circ \mathcal{T}(X) = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \left(\left([X] + \begin{bmatrix} -3 \\ -6 \end{bmatrix} \right) - \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

If you distribute across the parenthesis, multiply the constant vectors by the matrix, and collect terms, you'll find that these are both (!) equal to

$$\begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} [X] + \begin{bmatrix} -27/5 \\ 24/5 \end{bmatrix}$$

So with these particular choices, it doesn't matter if you do the reflection first and then the translation, or vice versa; your answers to #6 and #7 are the same. (Why does it turn out that way? Hint: $(-3, -6) = -3(1, 2)$ could serve as a direction indicator for the line, so the composition in either order gives the same glide reflection!)

7.9.13: Equation (7.14), which I developed in a slightly different way in class, tells us that the formula for a central inversion (i.e. a rotation by π) about a point C is $\mathcal{C}_C(X) = -X + 2C$ or, if $C = (h, k)$,

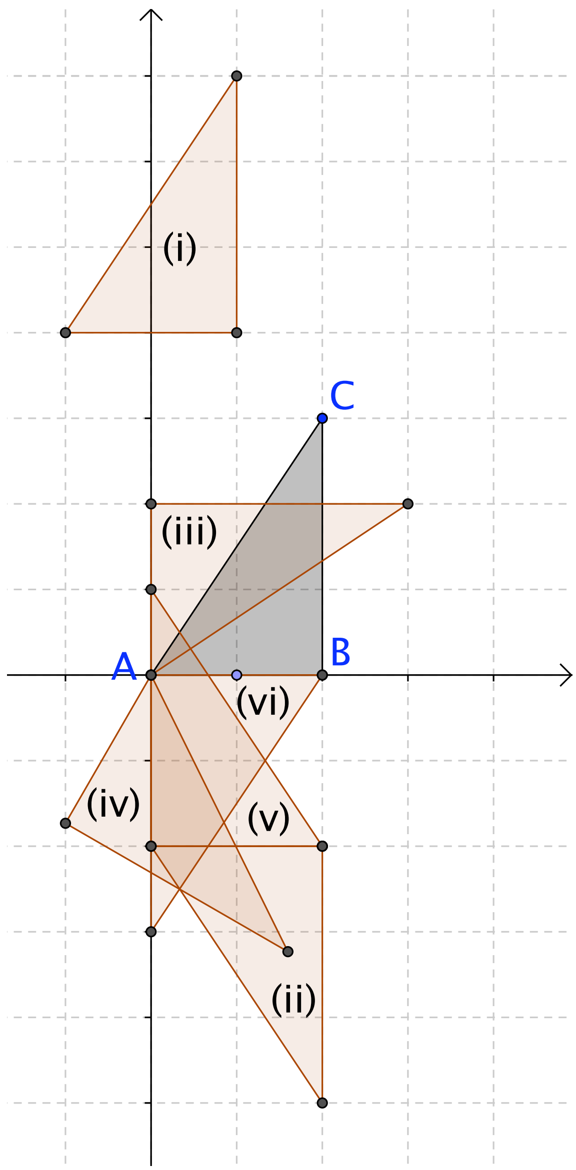
$$\mathcal{C}_C(x, y) = -(x, y) + 2(h, k) = (2h - x, 2k - y)$$

The only way $\mathcal{C}_C$ can leave (x, y) fixed (i.e. unchanged) is if

$$x = 2h - x$$
$$y = 2k - y$$

The only solution to this system is $x = h$ and $y = k$, i.e. $X = C$.

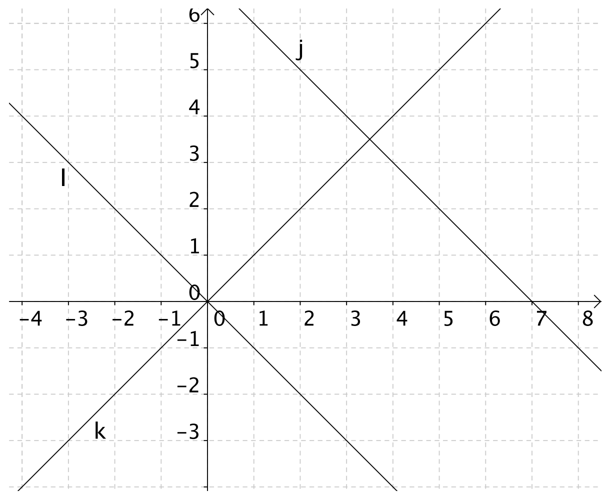
7.9.18: Here's a picture from GeoGebra showing the original $\triangle ABC$ and its images under the various isometries. Let me know if you have trouble matching the images up to the isometries, or calculating specific points, etc.



7.9.19: Any reflection is an involution, but the composition of two reflections can be a non-degenerate translation, which is not an involution.

7.9.30(iv,v): These two problem involve three lines, which I've named as follows and graphed with GeoGebra:

$$j : (2, 5) + s(3, -3)$$
$$k : \langle(-3, 3), X\rangle = 0$$
$$l : \langle(-3, -3), X\rangle = 0$$



There is a tough way to do these problems, and an easy way. The tough way is to construct the matrix formulas of the three reflections $\mathcal{M}_j$, $\mathcal{M}_k$, $\mathcal{M}_l$ across these three lines (similar to the work in #6 and #7 above), and then go through the tedium of writing out the compositions, multiplying matrices together, multiplying vectors by matrices, collecting terms, etc.

The easy way is to use our knowledge about the composition of reflections. In part (iv) we are working with j and k, lines which meet at a right angle at $C = (7/2, 7/2)$. (Verify this!) Hence either $\mathcal{M}_j \circ \mathcal{M}_k$ or $\mathcal{M}_k \circ \mathcal{M}_j$ will be a central inversion about the point C. (Technically you could argue that one will be a rotation by π and the other a rotation by $-\pi$, but the end result is the same.) The formula is given by

$$\mathcal{C}_C(X) = -X + 2C = -IX + 2C$$

where I is the 2×2 identitty matrix.

In part (v) we are working with j and l, lines which are parallel. The vector $U = (7/2, 7/2)$ is perpendicular to both and stretches exactly from $O \in l$ to $(7/2, 7/2) \in j$. (Check all of this!) The composition of two reflections across parallel lines is a translation in the direction from the first to the second, with a distance twice that between the lines. Thus:

$$\mathcal{M}_j \circ \mathcal{M}_l(X) = X + 2U$$
$$\mathcal{M}_l \circ \mathcal{M}_j(X) = X - 2U$$