Chapter 1 - Flythrough

This will be a fast review/intro - You're responsible for reading Chapter 1 (and 2) and talking to me if there's something you don't follow.

(Think: big ideas/terms, like "set," not the intricacies of, say, proof of Proposition 1.30...
Fundamentals / Vocabulary

$A$: for all, for every

$\exists$: there exists ($\exists!$: there exists a unique)

iff: if and only if, $\iff$

We won't use sets in much depth. Mostly:

$\mathbb{R} = \{\text{real #s}\} = \{x : x \in \mathbb{R}\} = \{x \mid x \in \mathbb{R}\}$

$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$

and esp. subsets of those
Recall “Abstract” Function Notation

\[ f: A \to B \]

- \( f(x) \): function value
- \( x \): input
- \( A \): domain, inputs
- \( B \): codomain (range, image)
  - set of possible outputs

**Example:**

\[ f: \mathbb{R} \to \mathbb{R} \quad \text{or} \quad \{ x \in \mathbb{R} : x \geq 0 \} \]

\[ x \mapsto x^2 \]

\[ f(x) = x^2 \]

**Definition:**

- \( f \) is injective (or one-to-one, 1:1) if any two different inputs are sent to different outputs: \( x \neq y \Rightarrow f(x) \neq f(y) \).

- \( f \) is surjective (onto) if every element of codomain is actual output: \( \forall b \in B \exists a \text{ such that } f(a) = b \).
## Functions Sheet

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Vectors, Points and Lines

A (2D) vector is an ordered pair of real #’s, \((a, b)\).

Common notations: \((a, b)\), \(\langle a, b \rangle\), \(\overrightarrow{ab}\) = \(\vec{a}\)

Our book: \(U = (u_1, u_2)\) \(X = (x_1, x_2)\)

Graphically, \(U\) is an arrow:

\[
\begin{array}{c}
\text{\(U\)} \\
\text{\(u_1\)} \\
\text{\(u_2\)}
\end{array}
\]

(here \(u_1, u_2 > 0\))

Virtually every vector concept has an algebraic definition and a geometric/axiomatic one.
Addition: \[ U + V = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2) \]

(scalar) multiplication: \[ cU = c(u_1, u_2) = (cu_1, cu_2) \]

Subtraction: \[ U - V = U + (-1)V \]

Linearly dependent: \[ \exists \ a, b \text{ such that } aU + bV = 0 \]
\[ a, b \neq 0 \]
\[ \mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle \]

**dot product**  \[ \mathbf{u} \cdot \mathbf{v} = (u_1, u_2) \cdot (v_1, v_2) \]

**inner product, scalar product**  \[ = u_1 v_1 + u_2 v_2 \]

\[ \neq (u_1, v_1, u_2 v_2) \]

**length/magnitude**  \[ \mathbf{u} \cdot \mathbf{u} = (u_1, u_2) \cdot (u_1, u_2) \]

\[ = u_1^2 + u_2^2 \]

\[ = \| \mathbf{u} \|^2 \]

\[ \| \mathbf{u} \| = \sqrt{\mathbf{u} \cdot \mathbf{u}} \]
If you learned vectors from Stewart's book...

Stewart make huge distinctions between points and vectors:

\[ \vec{u} = \langle 3, 2 \rangle = \overrightarrow{OP} \]

We won't. For us, vector and point are synonyms.

It's clear from context, and it makes life easier to do it this way. To wit:
Def: Given a point $P$ and non-zero vector $U$, the set

\[ l = \{ P + sU : s \in \mathbb{R} \} \]

is a line.

$U$ is direction indicator (dir'n vector). Points on $l$ are incident with $l$, and are collinear.

Ex: $( -1,2 ) + s(3,-4) \quad ( \frac{1}{2},0 )$ is on line ($s = \frac{1}{2}$)

$(5,6)$ is not. \[ -1 + 3s = 5 \]
\[ 2 - 4s = 6 \] no sol'n
Important Example. What’s $U$?

$U = Q - P$

want $P + U = Q$

$U = Q - P$
\[ P + s(Q-P) \]

\[ s \in (0, 1) \]

\[ \overrightarrow{PQ} = \overrightarrow{QP} \]

\[ \{ P + s(Q-P) \} \quad \{ Q + s(P-Q) \} \]

\[ \overrightarrow{PQ} \neq \overrightarrow{QP} \]

\[ \| X \| = \sqrt{X \cdot X} \]

\[ \text{length of } \overrightarrow{PQ} \text{ is } \| Q-P \| = \| P-Q \| \]

\[ = \sqrt{\langle Q-P, Q-P \rangle} \]

\[ = \sqrt{(Q-P) \cdot (Q-P)} \]

\( \text{day 1} \)
Warmup Problem (9/12/18)

**Recall**: \( U \cdot V = (u_1, u_2) \cdot (v_1, v_2) = u_1v_1 + u_2v_2 \)

**Prove**: The dot product is **commutative**: \( U \cdot V = V \cdot U \)

\[
U \cdot V = u_1v_1 + u_2v_2 = v_1u_1 + v_2u_2 = V \cdot U
\]

**Prove**: The dot product is **distributive**: \( U \cdot (V + W) = U \cdot V + U \cdot W \)

\[
U \cdot (V + W) = (u_1, u_2) \cdot (v_1+w_1, v_2+w_2)
\]

\[
= u_1(v_1+w_1) + u_2(v_2+w_2)
\]

\[
= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2
\]

\[
= (u_1v_1 + u_2v_2) + (u_1w_1 + u_2w_2)
\]

\[
= U \cdot V + U \cdot W
\]
Does the def of \{P+sU\} cover everything we expect?

- Can get segments, rays using restricted values of s.

- Two points form a line? Yes (wksheet)

**Prop 1.4** Two non-zero vectors are DI's of same line iff they're scalar multis of each other.

Let P \neq Q. Then \exists unique line \overrightarrow{PQ} incident with both, \overrightarrow{U} = Q-P is a DI of \overrightarrow{PQ}, and every DI of \overrightarrow{PQ} is difference of two pts on the line.
Other Forms

Ex. \((-1,2) + s(3,4) = (-1,2) + (3s,4s) = \left(\frac{-1+3s}{x}, \frac{2+4s}{y}\right)\)

\[
\begin{align*}
x &= 3s - 1 \\
y &= -4s + 2
\end{align*}
\]

\[
\begin{align*}
\frac{1}{3}(x+1) &= -\frac{1}{4}(y-2) \\
(y-2) &= -\frac{4}{3}(x+1) \quad \text{(Pt slope)}
\end{align*}
\]

\[
\begin{align*}
y &= 2 - \frac{4}{3}x - \frac{4}{3} \\
y &= -\frac{4}{3}x + \frac{2}{3} \quad \text{(slope int.)}
\end{align*}
\]
Def. Two lines $l, m$ are parallel, $l \parallel m$, if their DI's are II.

Prop 1.6 Lines $l = \{P + sU\}, m = \{Q + tV\}$

- $\cap$ in one pt if $U, V$ linearly independent \((\text{not } II)\)
- empty $\cap$ if $U \parallel V$ and $U \nparallel Q - P$
- same line if $U \parallel V$ and $U \parallel Q - P$
Quick Status Check: which of Euclid's Axioms work so far?

1. Given two pts, ∃ line containing them  yes
2. Lines can be extended indefinitely  yes
3. Given A, B, ∃ circle cent'd at A with radius AB.  yes
4. Right angles are all equal  x
5. || postulate  (yes, HW)
Perpendicularity / Orthogonality

**Def**: $U \perp V$ if $U \cdot V = 0$. Two lines are perpendicular if their DI's are $\perp$.

$\parallel$ and $\perp$ play important roles...

**Corollary 1.11**: If $l$ is a line and $P$ is a point, $\exists$ exactly one line incident with $P$ and $\parallel$ to $l$.

**Prop 1.15**: If $l$ is a line and $P$ is a point, $\exists$ exactly one line incident with $P$ and $\perp$ to $l$. 
Corollary 1.16. The set of vectors \( \perp \) to \( U = (u_1, u_2) \neq 0 \) consists of all multiples of \( (-u_2, u_1) \).

**Pf:** First, we see that \((u_1, u_2) \cdot (-u_2, u_1) = -u_1u_2 + u_2u_1 = 0\).

Now suppose \( V \perp U \), so that \( u_1v_1 + u_2v_2 = 0 \). We want to show \( V = c(-u_2, u_1) \) for some \( c \).
**Prop 1.15** If \( l \) is a line and \( P \) is a point, \( \exists \) exactly one line incident with \( P \) and \( l \) to \( l \).

**Proof** Suppose \( l: Q + sU, \ U \neq 0 \)

Then \( m: P + s(-u_2, u_1) \) is \( \perp l \).

Suppose \( \exists \) another such line \( P + sW \), so \( U \perp W \)

By prev. slide \( W = cV \) for some \( c \), so \( W \parallel V \).

By Prop 1.6, they are same line.
**Normal Form**

Given line \( l \), choose \( Y \in l \) and \( A \perp l \)
(i.e. \( A \perp U \), \( U \) any DI of \( l \)). Then

\[
l = \left\{ X : A \cdot (X - Y) = 0 \right\}
\]

\( \text{nl eqn of line} \)

\( \checkmark \) \( A, Y \) are constants; \( X = (x_1, x_2) \)
(\( = (x, y) \) )

**Ex**  \( Y = (3, 1) \), \( A = (-1, 2) \)

\[
\begin{align*}
(-1, 2) \cdot (x - (3, 1)) &= 0 \\
(-1, 2) \cdot ((x_1, x_2) - (3, 1)) &= 0 \\
(-1, 2) \cdot (x_1 - 3, x_2 - 1) &= 0 \\
- (x_1 - 3) + 2(x_2 - 1) &= 0 \\
(x_2 - 1) &= \frac{1}{2} (x_1 - 3) \\
y &= \frac{1}{2} x - \frac{1}{2}
\end{align*}
\]
An alternate version of normal form:

\[ A \cdot (x-y) = 0 \]
\[ A \cdot x - A \cdot y = 0 \]

\[ A \cdot x = A \cdot y = c \]
\[ A \cdot x = c \]

**Ex**

\[ A = (-1, 2), \ Y = (3, 1) \]

\[ (-1, 2) \cdot ((x, y) - (3, 1)) = 0 \]

\[ (-1, 2) \cdot (x, y) - (-1, 2) \cdot (3, 1) = 0 \]

\[ [-1, 2] \cdot (x, y) + 1 = 0 \]

\[ (-1, 2) \cdot x = -1 \]
Second Warmup Question:

Prove: \((cU) \cdot V = c (U \cdot V)\) \(= V \cdot (cU)\) etc...

Pf: \((cU) \cdot V = (cu_1, cu_2) \cdot (v_1, v_2)\)
\[= cu_1v_1 + cu_2v_2\]
\[= c (u_1v_1 + u_2v_2)\]
\[= c (U \cdot V)\]

Prove: \(\forall U, \| -U \| = \|U\|\)
\[\| -U \|^2 = (-U) \cdot (-U)\]
\[= [(-1)U] \cdot [(-1)U]\]
\[= (-1)^2 U \cdot U\]
\[= U \cdot U\]
\[= \|U\|^2\]
Betweeness

Def/Prop Let \( f(s) \) be eqn of line, \( f(s_1) = P, f(s_2) = Q \). Then \( R \) is between \( P, Q \) if \( \exists s_3, s_1 < s_3 < s_2, f(s_3) = R \)

Corollary 1.22 Given 3 pts on a line, one must be b/w other two.

See book for further corollaries with 4+ points
A line separates $\mathbb{R}^2$ into two “half planes.”

Clever Def \( P, Q \& l \) on opposite sides of \( l \) if \( \exists R \in l \) which is between them.
Prop 1.30 \[\text{Let } l : A \cdot (X-Y) = 0 \text{ (Yet, } A \perp l) \text{ and } P, Q \notin l. \text{ Then } P \text{ and } Q \text{ are on same/opposite side of } l \text{ if } A \cdot (P-Y), A \cdot (Q-Y) \text{ have same signs.}\]

⚠️ Book uses \(A \cdot X = c\), compares \(A \cdot P - c, A \cdot Q - c\).

⚠️ Would be “simple” (well, simpler) if we had

\[\vec{a} \cdot \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cos \theta > 0 \text{ for } \theta \in [0, \pi/2)\]
\[< 0 \text{ for } \theta \in (\pi/2, \pi]\]
Prop 1.30 Let \( l : A \cdot (x - y) = 0 \) \((y \in l, A \perp l)\) and \( P, Q \in l \). Then \( P \) and \( Q \) are on same/opposite side of \( l \) if \( A \cdot (P - y), A \cdot (Q - y) \) have same/opposite sign.

pf Let \( g(s) = A \cdot \left[ (P + s(Q - P)) - y \right] \) for \( 0 \leq s \leq 1 \)

line seg. \( PQ \)

\[ g(s) = 0 \text{ for some } s \iff \quad \overline{PQ} \text{ intersects } \overline{l} \text{ at pt } R \]
\[ (\Rightarrow P, Q \text{ opposite sides}) \]

\[ g(s) = A \cdot (P - y) + s \cdot A \cdot (Q - P) = \cdots = b + s \cdot c \quad (1 + 2s) \]

That's linear! min/max's at endpoints, can be 0 if +/- \( o \text{ or } -l \)

at endpoints

\[ g(0) = A \cdot (P - y) \]
\[ g(1) = A \cdot (Q - y) \]
Moving on to Chapter 2...

You read §2.1 on "Matrix Concepts. Other than matrix multiplication, all we'll need for now:

Lemma 2.1 Let $U, V \in \mathbb{R}^2$ be linearly independent. Then

$\forall x \in \mathbb{R}^2, \exists$ unique $a, b \in \mathbb{R}$ such that

$x = aU + bV$

linear comb'n

(see lemma for formulas for $a, b$. in particular...
Lemma 2.4 Let \( U, V \neq 0 \) in \( \mathbb{R}^2 \), with \( U \perp V \). Let \( X \in \mathbb{R}^2 \). Then

\[
X = \frac{X \cdot U}{\| U \|^2} U + \frac{X \cdot V}{\| V \|^2} V
\]

Corollary 2.5 (Same conditions) \[ \| X \|^2 = \frac{(X \cdot U)^2}{\| U \|^2} + \frac{(X \cdot V)^2}{\| V \|^2} \]
Distances and Inequalities

We defined \( \|x\| = \sqrt{x \cdot x} = \langle x, x \rangle^{\frac{1}{2}} \Rightarrow (\|x\|^2 = x \cdot x) \)

Also,

\[ \|Q-P\| = \text{dist from } P \text{ to } Q = \|P-Q\| = |\overrightarrow{PQ}| = |\overrightarrow{QP}| \]

\( \forall u, \|u\| = \|\overrightarrow{u}\| \leftrightarrow \text{warmup} \)

Def \( \overrightarrow{PQ} \sim \overrightarrow{RS} \) are congruent if \( |\overrightarrow{PQ}| = |\overrightarrow{RS}| \).

Prop Congruence of line segments is equivalence relation.
Quick Aside: Equivalence Relations

Examples of Relations

\( \mathbb{Z}, < : 3 < 4, 4 \not< 3, 5 < 3. \)  \( \mathbb{Z}, R : aRb \iff a^2 = b^2 \)

\( \mathbb{Z}, = : 3 = 3, 3 \neq 4 \)

A relation is an equivalence relation if it is...

1. reflexive: \( \forall x, x \sim x \)
2. symmetric: \( \forall x, y, \text{if } x \sim y \text{ then } y \sim x \)
3. transitive: \( \forall x, y, z, \text{ if } x \sim y \text{ and } y \sim z \text{ then } x \sim z. \)
You try: (I didn't type this up...)

Which of the following are equivalence relations?

- people, "have same birthday"  yes
- lines, \( \parallel \)  yes
- integers, \( \leq \)  not symm (\( 1 \leq 2, 2 \not\parallel 1 \))
- lines, \( \perp \)  not refl., trans
- \( \mathbb{R} \), aRb if \( a^2 = b^2 \)  yes
- \( \mathbb{R} \), \( \approx \) probably not [not trans?]
- L approx equal to.
Cauchy-Schwarz (-Bungakovsky) Inequality (Lemma 2.11)

\[ |u \cdot v| \leq \|u\| \cdot \|v\| \quad \text{with equality iff } u \parallel v. \]

1) Standard Pf

(done on board)

2) Quicker Pf using lin. alg. concepts from Chapter 2.

(done on board)
Lemma 2.12 (\(\Delta\) inequality):

(done on board)

(Restated in Prop 2.13 w/ line segments)
Thm 1.55 (Pythagorean) Let $A, B, C \in \mathbb{R}^2$ be distinct points.

(done on board)