Chapter 1 - Flythrough

This will be a fast review/intro — You're responsible for reading Chapter 1 (and 2) and talking to me if there's something you don't follow.

(Think: big ideas/terms, like “set,” not the intricacies of, say, proof of Proposition 1.30...)

* I'll post a review of solving systems of linear eqns using subst’n, elimination or matrix multiplication.
Fundamentals / Vocabulary

∀ : for every, for all

∃ : there exists (∃! : there exists a unique)

iff : iff and only if, ⇔

We won't use sets in much depth. Mostly:

\[ R = \text{real numbers} = \{ x : x \in \mathbb{R} \} \]

\[ \mathbb{R}^2 = \{ (x, y) : x, y \in \mathbb{R} \} \]

and esp. subsets of those
Recall “Abstract” Function Notation

\[ f: A \rightarrow B \]

- \( A \): domain
- \( B \): codomain (image, range)
  - target (space)

\[ x \mapsto f(x) \]

“maps to”

**Example:**

\( f: \mathbb{R} \rightarrow \mathbb{R} \) or \( \{ x \in \mathbb{R} : x \geq 0 \} \)

\[ x \mapsto x^2 \]

**Definition:**

- **Injective** or **One-to-One** (1:1): two elts in domain sent to different outputs.

- **Surjective** or **Onto**: every elt of codomain is hit.
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Vectors, Points and Lines

A (2D) vector is an ordered pair of real #’s, \((a, b)\).

Common notations: \(\langle a, b \rangle, \overrightarrow{a, b}\)

Our book: \(\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2)\)

Graphically, \(\mathbf{U}\) is an arrow:

- \(\mathbf{u}_1\) (here \(\mathbf{u}_1, \mathbf{u}_2 > 0\))

Virtually every vector concept has an algebraic definition/interpretation and a geometric/axiomatic one.
**Addition**

\[ U + V = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2) \]

**Subtraction**

\[ U - V = U + (-1)V = (u_1 - v_1, u_2 - v_2) \]

**Linearly dependent**

\[ U = cV \text{ or } V = cU \]

\[ \exists a, b \text{ such that } aU + bV = 0 \text{ (}= (0,0)) \]

\[ a, b \text{ not both 0.} \]
dot product: $\mathbf{U} \cdot \mathbf{V} = (u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2 
\neq (u_1 v_1, u_2 v_2)$

$\mathbf{U} \cdot \mathbf{V} = \langle \mathbf{u}, \mathbf{v} \rangle$

length/magnitude: $||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2}$
If you learned vectors from Stewart's book...

Stewart make huge distinctions between points and vectors:

![Diagram]

\[ \vec{u} = \langle 3, 2 \rangle = \overrightarrow{OP} \]

We won't. For us, vector and point are synonyms.

It's clear from context, and it makes life easier to do it this way. To wit:
Def. Given a point \( P \) and non-zero vector \( U \), the set

\[ l = \{ P + sU : s \in \mathbb{R} \} \]

is a line.

\( U \) is a direction indicator (dir. vector). Points on \( l \) are "incident" with line.

\( P_1, P_2, \ldots, P_n \) are collinear if \( \exists \) line incident w/ all of them.

Ex. \((-1, 2) + s(3, -4)\) \((\frac{1}{2}, 0)\) is on line \((s = \frac{1}{2})\)

\((5, 6)\) is not. \[-1 + 3s = s\]
\[2 - 4s = 6\] no soln.
\[ P + s(Q-P) \]

\[ Q - P = u \]

\[ s \in (0,1) \]

\[ \vec{PQ} = -\vec{QP} \]

\[ \{ P + s(Q-P) \} \quad \{ Q + s(P-Q) \} \]

\[ \vec{PQ} = \vec{QP} \]

\[ \vec{PQ} \neq \vec{QP} \]

\[ \| X \| = \sqrt{X \cdot X} \]

Length of \( \vec{PQ} \) is \( \| Q-P \| = \| P-Q \| \)

\[ = \sqrt{(Q-P) \cdot (Q-P)} \]
Does this def. cover everything we expect?

- Can get segments, rays using restricted values of s.

- Two points form a line? Yes (wksheet)

Prop. 1.4 two non-zero vectors are DI's of same line
iff they're scalar multis of each other.

Let $P \neq Q$. Then there's a unique line $\overrightarrow{PQ}$ incident with both, $U = \overrightarrow{Q-P}$ is a DI of $\overrightarrow{PQ}$, and every DI of $\overrightarrow{PQ}$ is difference of two pts on the line.

- Slope of $P+sU$ is $\frac{u_2}{u_1}$, if $u_1 \neq 0$. 
other forms: eliminate parameter

\[ \text{Ex } (-1,2) + s(3,-4) - (-1+3s, 2-4s) \]
\[ \frac{x(t)}{x(t)} \quad \frac{y(t)}{y(t)} \]

\[ x = 3s - 1 \quad \Rightarrow \quad s = \frac{1}{3}(x+1) \]
\[ y = -4s + 2 \quad \Rightarrow \quad s = \frac{1}{4}(y-2) \]

\[ \frac{1}{3}(x+1) = - \frac{1}{4}(y-2) \]

\[ 4x + y = -3y + 6 \quad \text{solve for } y... \]

\[ 3y = -4x + 2 \]
\[ y = -\frac{4}{3}x + \frac{2}{3} \]
Def Two lines \( l, m \) are parallel, \( l \parallel m \) if DI's are \( II \).

Prop 1.6 Lines \( l = \{ P + s U \}, m = \{ Q + t V \} \):

- \( \cap \) in one point if \( U, V \) lin. indep (not \( II \))
- empty \( \cap \) in if \( U \parallel V \) and \( U \parallel Q - P \)
- are same line if \( U \parallel V \) and \( U \parallel Q - P \)
Recall: \( U \cdot V = (u_1, u_2) \cdot (v_1, v_2) = u_1v_1 + u_2v_2 \)

Prove: The dot product is commutative: \( U \cdot V = V \cdot U \)

\[ U \cdot V = u_1v_1 + u_2v_2 = v_1u_1 + v_2u_2 = V \cdot U \]

Prove: The dot product is distributive: \( U \cdot (V+W) = U \cdot V + U \cdot W \)

\[ U \cdot (V+W) = (u_1, u_2) \cdot (v_1+w_1, v_2+w_2) \]

\[ = u_1(v_1+w_1) + u_2(v_2+w_2) \]

\[ = u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 \]

\[ = (u_1v_1+u_2v_2) + (u_1w_1+u_2w_2) \]

\[ = U \cdot V + U \cdot W \]
Quick Status Check: which of Euclid’s Axioms work so far?

1. Given two pts, ∃ line containing them ✓
2. Lines can be extended indefinitely ✓
3. Given A, B, ∃ circle cent’ed at A with radius AB. ✓
   \[ r = ||B - A|| \]
   \[ C = \{ x : ||x - A|| = r \} \]
4. Right angles are all equal ❌
5. || postulate (✓ HW)
**Perpendicularity / Orthogonality**

**Def** \( U \perp V \) if \( U \cdot V = u_1v_1 + u_2v_2 = 0 \). Two lines \( l, m \) are perpendicular if they have \( \perp \) Direction Indicators.

\[ \parallel \text{ and } \perp \text{ play important roles...} \]

**Corollary 1.11** If \( \ell \) is a line and \( P \) is a point, \( \exists \) exactly one line incident with \( P \) and \( \parallel \) to \( l \).

**Prop 1.15** If \( \ell \) is a line and \( P \) is a point, \( \exists \) exactly one line incident with \( P \) and \( \perp \) to \( l \).
Prop 1.15 If \( l \) is a line and \( P \) is a point, there is exactly one line incident with \( P \) and \( l \) to \( l \).

If \( \langle l, P \rangle \) is \( \mathbb{R} + s \mathbf{u}, \mathbf{u} \neq 0 \)

Claim: \( \mathbf{u} = (u_1, u_2) \Rightarrow \mathbf{v} = (-u_2, u_1) \perp \mathbf{u} \)

Now show any vector \( \mathbf{v} \perp \mathbf{u} \) is scalar multiple of this.

Case 1 \( \mathbf{u} = (u_1, u_2), u_1 = 0 \Rightarrow \mathbf{v} = (-u_2, 0) \)

\[ \mathbf{u} \cdot \mathbf{v} = (0, u_2) \cdot (-u_2, 0) = 0 + 0 = 0 \]

Case 2 \( u_2 = 0 \)

Case 3 in general, \( \mathbf{u} \cdot \mathbf{v} = (u_1, u_2) \cdot (-u_2, u_1) = -(u_1 u_2 + u_1 u_2) = 0 \)

\( \mathbf{m}: \mathbf{P} + s \mathbf{v} \) incident u/p, \( \perp l \).
Normal Form

Given line $l$, choose $Y \in l$ and $A \perp l$ (i.e. $A \parallel U$, $U$ any DI of $l$). Then

$$l = \{ X : A \cdot (X-Y) = 0 \}$$

nl eqn of line

Remember $A, Y$ fixed; $X = (x_1, x_2) \ (= (x, y_1))$ is variable. If $||A|| = 1$, this is "special" nl eqn.

Ex $Y = (3, 1), A = (-1, 2)$

$$(-1, 2) \cdot (X-Y) = (-1, 2) \cdot ((x_1, x_2) - (3, 1))$$
$$= (-1, 2) \cdot (x_1-3, x_2-1)$$
$$= -x_1 + 3 + 2x_2 - 2 = 0$$

$$2x_2 = x_1 - 1$$
$$x_2 = \frac{1}{2} x_1 - \frac{1}{2}$$

$$y = \frac{1}{2} x - \frac{1}{2}$$
An alternate version of normal form:

\[ A \cdot (x - y) = 0 \]

\[ A \cdot x - A \cdot y = 0 \]

\[ A \cdot (x, y) - c = 0 \]

\[ A \cdot x = c \quad \text{[} A \cdot (x_1, x_2) = c \text{]} \]

\[ \text{Ex:} \quad A = (-1, 2), \quad Y = (3, 0) \]

\[ A \cdot (x - y) = 0 \]

\[ A \cdot x - A \cdot y = 0 \quad \Rightarrow \quad A \cdot x + 1 = 0 \]

\[ = -1 \quad A \cdot x = -1 \]
**Betweeness**

**Def/Prop** Let \( f(s) \) be eqn of line, \( f(s_1) = P, f(s_2) = Q \). Then \( R \) is **between** \( P, Q \) if \( \exists \ s_3, \ s_1 < s_3 < s_2, f(s_3) = R \).

\[
\begin{align*}
R &= f(s_3) \\
P &= f(s_1) \\
Q &= f(s_2)
\end{align*}
\]

**Corollary 1.22** Given 3 pts on a line, one must be b/w other two.

See book for further corollaries with 4+ points
A line separates $\mathbb{R}^2$ into two “half planes.”

Clever Def: $P, Q \notin l$ on opposite sides of $l$ if $\exists R \in l$ between them. Otherwise, they’re on the same side.
Prop 1.30: Let $l: A \cdot (X-Y) = 0$ (i.e., $A \perp l$) and $P, Q \notin l$. Then $P$ and $Q$ are on the same/opposite side of $l$ if $A \cdot (P-Y), \ A \cdot (Q-Y)$ have same signs.

⚠ Book uses $A \cdot X = c$, compares $A \cdot P - c, A \cdot Q - c$.

⚠⚠ Would be “simple” (well, simpler) if we had

$$\hat{a} \cdot \hat{b} = ||\hat{a}|| \cdot ||\hat{b}|| \cos \theta > 0 \text{ for } \theta \in [0, \pi/2)$$
$$< 0 \text{ for } \theta \in (\pi/2, \pi]$$
Prop 1.30 Let \( l : A \cdot (x-y) = 0 \) \((Y \in l, A \perp l)\) and \( P, Q \in l \). Then \( P \) and \( Q \) are on same/opposite side of \( l \) if \( A \cdot (P-Y), A \cdot (Q-Y) \) have same/opposite sign.

Pf Let \( g(s) = A \cdot (P + s(Q-P) - Y) \) for \( 0 \leq s \leq 1 \)

The line segment \( PQ \)

\( g(s) = 0 \) for some \( s \) iff \( PQ \) intersects \( l \) at pt \( R \)

\((=) P, Q \) opposite sides)

Now \( g(s) = A \cdot (P-Y) + sA \cdot (Q-P) = b + ms = ms+b \)

That's linear, so min/max values @ endpoints, \( g(s)=0 \) only if \( g(0), g(1) \) have diff. signs.

\( g(0) = A \cdot (P-Y), g(1) = A \cdot (Q-Y). \)
Second Warmup Question:

Prove: \((cU) \cdot V = c(U \cdot V)\)

\[
\text{Pf: } (cU) \cdot V = (cu_1, cu_2) \cdot (v_1, v_2)
\]

\[
= cu_1v_1 + cu_2v_2
\]

\[
= c(u_1v_1 + u_2v_2)
\]

\[
= c(U \cdot V)
\]
Moving on to Chapter 2...

You read §2.1 on “Matrix Concepts. Other than matrix multiplication, all we’ll need for now:

Lemma 2.1 Let $U, V \in \mathbb{R}^2$ be linearly independent. Then for all $X \in \mathbb{R}^3$, there exist unique $a, b \in \mathbb{R}$ such that

$$X = aU + bV$$

“coords $(a, b)$”

linear combination of $U, V$.

(See lemma for formulas for $a, b$. in particular...)
Lemma 2.4 Let $U, V \neq 0$ in $\mathbb{R}^2$, with $U \perp V$. Let $X \in \mathbb{R}^2$. Then

$$X = \frac{X \cdot U}{\|U\|^2} U + \frac{X \cdot V}{\|V\|^2} V$$

Corollary 2.5 (Same conditions) $\|X\|^2 = \frac{(X \cdot U)^2}{\|U\|^2} + \frac{(X \cdot V)^2}{\|V\|^2}$
Distances and Inequalities

We defined \( \|x\| = \sqrt{x \cdot x} = \langle x, x \rangle^{\frac{1}{2}} \)

Also,

\[
\|Q - P\| = \text{dist from } P \text{ to } Q = \|P - Q\| = |\overline{PQ}| = |\overline{QP}|
\]

\( \forall u, \|u\| = \| - u\| \)

Def \( \overline{PQ} \sim \overline{RS} \) are congruent if \( |\overline{PQ}| = |\overline{RS}| \).

Def Congruence of line segments is equivalence relation.
Quick Aside: Equivalence Relations

Examples of Relations

\[ \mathbb{Z}, < : 3 < 4, 5 \neq 1 \quad R, \ a R b \text{ if } a^2 = b^2 \]

\[ R, = : 3 = 3, 3 \neq 4 \]

A rel'n is an equivalence rel'n if it is...

1. reflexive: \( \forall x, \ x R x \)
2. symmetric: \( \forall x, y, \text{ if } x R y \text{ then } y R x. \)
3. transitive: \( \forall x, y, z, \text{ if } x R y \text{ and } y R z, \text{ then } x R z \)
You try: (I didn’t type this up...)

Which of the following are equivalence relations?

- people, “have same birthday” yes
- integers, ≤ x not Symm
- lines, || yes integers, ≤ x not Symm
- lines, ⊥ x not reflexive or trans. IR, aRb if a²=b² yes
- L approx equal to.
Cauchy-Schwarz (Bunyakovsky) Inequality (Lemma 2.11)

\[ |U \cdot V| \leq \|U\| \cdot \|V\| \text{ with equality iff } U \parallel V. \]

1. Standard Pf

2. Quicker Pf using lin. alg. concepts from Chapter 2.

(These were done on board)

end day 2.
Lemma 2.12 (\(\triangle\) inequality):

(done on board)

(Re-stated in Prop 2.13 w/ line segments)
Thm 1.55 (Pythagorean) Let $A, B, C \in \mathbb{R}^2$ be distinct points.

\( \text{(done on board)} \)