This is all in the book, but presented in a different order or with a different viewpoint/approach (e.g. the formula for reflections). So follow class notes instead.

We know from Chapter 4: \( U : \mathbb{R}^2 \to \mathbb{R}^2 \) is an isometry iff \( U(x) = Hx + p, \ M \in \{ R_0, F_0 \} \)

From "Useful Facts": Let \( U = \begin{bmatrix} \cos \theta \\ -\sin \theta \end{bmatrix}, \ V = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \)

Then \( F_0 U = U, \ F_0 V = -V \)

\( F_0 (kU) = kU, \ F_0 (kV) = -kV \)

**Overriding Questions**: How many isometries are there? What are they? How do we know that list is complete?

Let's start with a few examples
Identity \( \Phi(X) = X \) \((= R_0 X + O)\)

Translation \( T_V (X) = X + V \) \((= T X + V = R_0 X + V)\)

(We saw those \( T \) in Chapter 4.)

Rotation by \( \Theta \) about \( C \) seems hard in general - so reduce to a previously solved problem! By construction, \( R_\Theta \) rotates \( \mathbb{R}^3 \) by \( \Theta \) about the origin.

Let's do this in steps!

1. Move \( C \) to origin: \( X-C \)
2. Rotate by \( \Theta \) about \( O \): \( R_\Theta(X-C) \)
3. Move \( O \) back to \( C \): \( R_\Theta(X-C) + C \)

\[ R_{\Theta,C}(X) = R_\Theta(X-C) + C \]

(Recall, from how composition of isometries is an isometry.)
Reflections: Also tricky. Let's do stages: refl'n across x-axis, across a line through O, across an y line.

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx + dy \\ cx + dx \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]

1. Rotate by \( -\theta \) 
2. Reflect across x-axis. \( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_\theta X \)
3. Rotate back: \( R_\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_\theta X \)

Note \( R_\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_\theta = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & -2\sin \theta \cos \theta \\ 2\sin \theta \cos \theta & \sin^2 \theta + \cos^2 \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{1}. \)

Thus \( T(X) = F_\theta X \) reflects \( \mathbb{R}^2 \) across line which forms angle of \( \theta \) w/ x-axis.

General Refl'n across \( l \)

(which contains a pt \( P \), forms angle of \( \theta \) w/ horizontal)

1. Move \( P \) to origin: \( (X-P) \)
2. Reflect: \( F_\theta(X-P) \)
3. Move \( O \) back to \( P \): \( F_\theta(X-P) + P \)
Ex: Get a pt e Quad II, \( u = (3, 4) \)

\[
\begin{align*}
\csc \theta &= \csc \theta - \tan \theta = \frac{9}{\sqrt{5}} - \frac{14}{\sqrt{5}} = -\frac{7}{\sqrt{5}} \\
\sec \theta &= 2 \cos \theta = 2 \cdot \frac{3}{\sqrt{5}} \cdot \frac{4}{\sqrt{5}} = \frac{24}{\sqrt{5}}
\end{align*}
\]

\[M_e(X) = \left[ -\frac{30}{\sqrt{5}}, \frac{24}{\sqrt{5}} \right] \left[ \begin{array}{c} x \\ y \end{array} \right] - \left[ \begin{array}{c} 7 \\ 7 \end{array} \right] + \left[ \begin{array}{c} 7 \\ 7 \end{array} \right]
\]

⚠️ Our formula for \( M_e \) seems to depend on arbitrary choice of P&L. Uh-oh (ex with other pt above on \( \delta \)).

Prop: Let \( P, Q \in \mathcal{L} \), which forms angle of \( \theta \) w/ the horizontal. Then

\[F_\theta(X - P) + P = F_\theta(X - Q) = Q\]

(And thus we can use any pt on \( \delta \) in formula for \( M_e \).)

Pf: By "useful facts," \( F_\theta(Q - P) = Q - P \)

**Method 1**

\[
\begin{align*}
F_\theta(X - P) + P &= F_\theta(X - P) - F_\theta P + P \\
F_\theta(X - Q) + Q &= F_\theta(X - Q) - F_\theta Q + Q
\end{align*}
\]

equal if \(-F_\theta P + P = -F_\theta Q + Q\)

\[F_\theta(Q - P) = Q - P \checkmark \]

**Method 2**

\[
\begin{align*}
F_\theta(Q - P) &= Q - P \\
F_\theta(X - X + Q - P) &= Q - P \\
F_\theta(X - P) - F_\theta(X - Q) &= Q - P \\
F_\theta(X - P) + P &= F_\theta(X - Q) + Q
\end{align*}
\]

⚠️ Two-sided pts

**Method 3**

\[
\begin{align*}
F_\theta(X - P) + P &= F_\theta(X - Q + Q - P) \\
&= F_\theta(X - Q) + F_\theta(Q - P) + P \\
&= F_\theta(X - Q) + Q - P + P \\
&= F_\theta(X - Q) + Q
\end{align*}
\]
**Prop** \( R_{\theta,c} \circ R_{\theta,c} = R_{\theta + \theta, c}(x) \)

**Pf**
\[
R_{\theta} \left( [R_{\theta}(x)] + c \right) + c = R_{\theta} R_{\theta}(x-c) + c = R_{\theta + \theta}(x-s) + c
\]

**Prop** \( m_{\alpha} m_{\beta} = m_{\alpha + \beta} \) (Remark: if \( U \circ U = \text{id} \), \( U \) is an involution)

**Pf**
\[
F_{\theta} \left( [F_{\theta}(x-p)] - p \right) + p = F_{\theta} F_{\theta}(x-p) + p \quad (F_{\theta} F_{\theta} = \text{id} - \text{useful facts})
\]

\[
= x - p + p = x
\]

**Prop** Let \( d \parallel k \) as shown. Then

\[
M_{k} \circ M_{d}(x) = f_{d, \theta}(x).
\]

**Pf**
\[
M_{k}(M_{d}(x)) = M_{k} \left( F_{\theta}(x-p) + p \right)
\]
\[
= F_{\theta} \left( (F_{\theta}(x-p) + p) - Q \right) + Q
\]
\[
= F_{\theta} F_{\theta}(x-p) - F_{\theta}(Q - P) + Q
\]
\[
= (x-p) + (Q-P) + Q
\]
\[
= x + Q - P = k \left[ \frac{-\sin \theta}{\cos \theta} \right]
\]

\[
V = Q - P = k \left[ \frac{-\sin \theta}{\cos \theta} \right]
\]
Prop If \( l \cap k \in c^3 \), then

\[
M_k \circ M_l = R(\varphi, \theta), c
\]

where \( l, k \) form angles of \( \varphi, \theta \) with the horizontal.

Remark Alternatively, the rotation is by twice the angle from \( l \) to \( k \). (In a counterclockwise direction— we now have positive, negative angles after Chapter 5!)

\[
\begin{align*}
M_k \circ M_l(x) &= M_k(F_0(x-c) + c) \\
&= F_\varphi(F_\varphi(F_\varphi(x) - c) + c) - c + c \\
&= F_\varphi F_\varphi(x - c) + c \\
&= R_{2(\varphi-\theta)} (\text{"useful facts"}) \\
&= R_{2(\varphi-\theta)}(x-c) + c \\
&= R_{2(\varphi-\theta), c}(x).
\end{align*}
\]
Back to rotations. What about $R_{\pi, b} \circ R_{\pi, c}$ with $C \neq D$?

**Prop** $R_{\pi, b} \circ R_{\pi, c} = T_{2(b-c)}$. (GeoGebra Demo.)

**Pf** Note $R_{\pi} = \begin{bmatrix} c & \pi \\ -\pi & c \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $R_{\pi} x = R_{\pi} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$

$R_{\pi, b} (R_{\pi, c}(x)) = R_{\pi} (R_{\pi}(x-c) + C)$

$= R_{\pi} (C - x + C)$

$= R_{\pi} \begin{bmatrix} 2c - x \\ -D \end{bmatrix} + D$

$= x - 2c + D + D$

$= x + 2(D - c)$.

General case (where angles aren't both $\pi$) is harder!

*(Turn for the worse activity)*

Ok — rather than test every possible combination, we need a systematic approach. The key is:

**Thm 6.11** Every isometry can be expressed (constructed) as the composition of $\leq 3$ reflexes. "Pf" : Lab 3.
Remark: Thus we "just" need to figure out all possibilities for $n=1, 2, 3$ reflexions.

\begin{align*}
n=1 & \quad M_l \text{ reflexion} \\
n=2 & \quad M_k \circ M_l \text{ is either: } \\
 & \quad \circ T_{2l} \text{ if } l \parallel k \\
 & \quad \circ R_{20,c} \text{ if } k \not\parallel l \text{ and } k = \text{rc3} \\
\end{align*}

Special cases: \(l \parallel l\): \(M_l \circ M_l = \chi(x) = T_{2l}(x) = R_{0,c}(x)\)

\(l \perp k\): \(M_k \circ M_l = C_c(x) \text{ 'central inversion'} = R_{0,c}(x)\)

\(n=3\) \(M_k \circ M_l \circ M_m = \ldots ?\)

\(\exists\) new possibility!

**Def:** Given \(u \parallel l\), \(G(x) = G(x) = M_l \circ T_{2l}(x)\) is a glide reflexion.

\[Ex \quad \rightarrow \quad \circ \circ \quad \circ \quad \circ \quad l\]
Proof: Given \( U \perp \ell \), \( M_u \circ T_u = T_u \circ M_u \).

(So we can do translation/reflex in a glide reflex in either order.)

\[
\text{Pf: } M_u \circ T_u(x) = F_\theta([x + u] - P) + P
\]

\[
= F_\theta x + F_\theta u - F_\theta P + P
\]

\[
= (F_\theta (x - P) + P) + u
\]

\[
= T_u(M_u(x))
\]

(Glide Reflex Sheet)

⚠️ How do we know \( g(x) \) isn't a reflex in the first?

Key turns out to be fixed pts.

Def: A fixed pt of a fn \( f: A \rightarrow A \) is a pt \( a \in A \) s.t. \( f(a) = a \).

Ex: \( f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 6 - 2x \). If \( f(x) = x \), then \( 6 - 2x = x \Rightarrow x = 2 \).

What about translations, rotations and reflections?

Prop: If \( U \neq 0 \), \( T_u \) has no fixed pts.

\[
\text{Pf: if } x + u = x, \ u = x - x = 0
\]
Prop. Only fixed pt of $R_{\theta,0}(X)$ is $X=0$.

\[ R_{\theta,0}(X) = RX = X \Rightarrow R_{\theta}X = X \]

\[ (R_{\theta}-I)X = 0 \]

$A$ is invertible. (check)

\[ \Rightarrow X = A^{-1}0 = 0. \]

Prop. Only fixed pt of $R_{\theta,c}(X)$ is $X=c$ $(\theta \neq 2\pi n)$

\[ R_{\theta,c}(X) = X \Rightarrow R_{\theta}(X-c)+c = X \]

Then $R_{\theta}(X-c) = (X-c)$

\[ \Rightarrow X-c = 0 \quad \text{so} \quad X = c. \]

Prop. Fixed pts of $M_{\theta}(X)$ are pts on $l$

If $X = F_{\theta}(X-P)+P$, then

\[ F_{\theta}(X-P) = (X-P) \]

We've seen (you check) fixed pts/vectors of $F_{\theta}$ are those $l$, so $X-P$ is $\perp$ of $l$ and $X = P + (X-P) \in l$. 
Prop A glide reflex \( G(x) \) has no fixed pts.

Pt \( G(x) = \mathcal{J}_u(M_e(x)) = F_0(x-p) + p + u. \)

Suppose \( F_0(x-p) + p + u = x \), so

\[ F_0(x-p) = (x-p) + u. \quad (1) \]

Null by \( F_0 \)

\[ F_0F_0(x-p) = F_0(x-p) + u \quad (F_0u = u) \]

\[ (x-p) - u = F_0(x-p) \quad (2) \]

So, if \( G(x) \) has fixed pt, (1) and (2) both true:

\[ F_0(x-p) = (x-p) + u = (x-p) - u \]

i.e. \( u = -u \Rightarrow u = 0 \Rightarrow \text{G a reflex} \).

(or if \( u = 0 \), \( G(x) \) not a glide reflex)
Prop  $G(X)$ is not a translin, rotlin or reflin. It's a "new" kind of isonicity.

Pf: $G(X) = T_u(M_e(X)) = F_0(X-P) + Pru$

$= F_0X + (stuff)$

Thus the matrix for $G(X)$ is $F_0$, so it's not a translin or rotlin.

By above, $G(X)$ has no fixed pts, so not a reflin.

Great - we now have 4 isoms: translins, rotins, reflins and glide reflins. Are there any others I can get with 3 reflins?

Consider $M_k \circ M_e \circ M_m$

either of these compositions of 2 reflins can be rewritten as a translin or rotlin - or maybe even $r((X)) = T_{0c}(X) = R_{0c}(X)$.

So we have to consider 4 possibilities:
(1) transl'n o refl'n
(2) refl'n o transl'n
(3) rot'n o refl'n
(4) refl'n o rot'n.

already know are GR's
if transl'n \parallel mirror, we'll
have to consider other cases
we can rewrite as compositions
of transl'n and refl'n (!!!), so
we don't have to worry about
(3), (4) !!

\textbf{Ex.} Consider (4), \( M_\theta \circ R_{\theta, C} \) (\( = M_\theta \circ M_k \circ M_m \))

\( R_{\theta, C} \) can be constructed by reflecting across
any two lines intersecting at \( C \), forming angle
of \( \theta/2 \) from \( 1^{st} \) line to \( 2^{nd} \). Let's choose
so that \( 2^{nd} \) line is \( \parallel \ell \):

\[
M_\theta \circ R_{\theta, C} = M_\theta (M_k \circ M_m)
= (M_\theta \circ M_k) \circ M_m
= F_{\theta, C} \circ M_m
\]

which is case (1) above.

Similarly, (3) can be rewritten as (2).
Thus, we only need to check \( T_u \circ M_u \) and \( M_u \circ T_u \) to see if we get any new isometries.

**Triple Reflection Sheet**

1st Case \( u \perp l \) \( \uparrow \)

**Prop.** Let \( u \perp l \). Then \( T_u \circ M_u = M_k \), where \( k = T_{u/2}(l) \) (i.e. \( k \) is the line \( l \), translated by \( u/2 \)).

**Pf.** Let \( p \perp l \).

\[
T_{u/2}(x) = T_{f_0}(x-p) + p
\]

\[
= f_0(x-p) + p + \frac{1}{2}u + \frac{1}{2}u
\]

\[
= f_0(x-p) + p - \frac{1}{2}f_0u + \frac{1}{2}u
\]

\[
= f_0(x-p - \frac{1}{2}u) + p + \frac{1}{2}u
\]

\[
= f_0(x - Q) + Q, \quad Q = p + \frac{1}{2}u
\]

\[
= M_k(x), \quad \text{for} \quad k = T_{u/2}(l).
\]

You check: \( M_u \circ T_u \) (other orders) also a reflex.
2nd Case: \( \mathbf{U} \) not \( \| \| \) or \( \perp \ell \) (General Case)

Write \( \mathbf{U} = \mathbf{A} + \mathbf{B} \), where \( \mathbf{A} \perp \ell \), \( \mathbf{B} \parallel \ell \).

Let \( P \in \ell \), so \( \mathbf{M}_\ell(x) = \mathbf{F}_\ell(x-P) + P \).

We know \( \mathbf{F}_\ell \mathbf{A} = -\mathbf{A} \), \( \mathbf{F}_\ell \mathbf{B} = \mathbf{B} \).

Let's check \( \mathcal{T}_u \circ \mathcal{M}_\ell(x) \), which is

\[
\mathbf{F}_\ell(x-P) + \mathbf{U} = \mathbf{F}_\ell(x-P) + P + \frac{1}{2} \mathbf{A} + \frac{1}{2} \mathbf{A} + \mathbf{B}
\]

\[
= \mathbf{F}_\ell(x-P) + P + \frac{1}{2} \mathbf{A} - \frac{1}{2} \mathbf{F}_\ell \mathbf{A} + \mathbf{F}_\ell \mathbf{B}
\]

\[
= \mathbf{F}_\ell \left( x - P - \frac{1}{2} \mathbf{A} + \mathbf{B} \right) + P + \frac{1}{2} \mathbf{A} \quad \mathbf{Q} = P + \frac{1}{2} \mathbf{A}.
\]

\[
= \mathbf{F}_\ell \left( \mathcal{T}_\ell (x) \right) + \mathbf{Q}
\]

\[
= \mathcal{M}_k \left( \mathcal{T}_\ell (x) \right) = \mathcal{G}(x) \text{ mirror } k = \mathcal{T}_k(x)
\]

glide \( \ell \).

We've proven:

Prop with above setup, \( \mathcal{T}_u \circ \mathcal{M} = \mathcal{G}(x) \) with

You check: \( \mathcal{M} \circ \mathcal{T}_u \) is also glide reflin.
We have (finally!) exhausted all possibilities, and have proven:

**Theorem.** The only possible isometries of \( \mathbb{R}^2 \) are:

- \( \mathcal{I}(X) \), the identity, orbit preserving, involution, fixed pts = \( \mathbb{R}^2 \), comp' of 0 or 2 reflns (cl: \( M_0 \) or \( 2 \mathcal{A} \) or \( \mathcal{A} \))

- \( \mathcal{M}_X \), refln across \( \ell \) orbit reversing, involution, fixed pts = \( \ell \)

- \( R(\theta) \) rot in by \( \theta \) about \( \mathcal{C} \) orbit preserving, not involvin (unless \( \theta = 0, \pi \)), fixed pts = \{ \mathcal{C} \} (unless \( \theta = 0 \)). Comp' of 2 reflns.
  
  *Special* rotins: \( R_0, c(X) = c(X) \), \( R_\frac{\pi}{2} c(X) = c(X) = 2 \mathcal{C} - X \) (involvin)
  
  rotins by \( \theta \neq 0, \pi \) called non-special.

- \( T_\mathcal{U}(X) \), translin by \( \mathcal{U} \) orbit preserving, non-involvin, no fixed pts unless \( \mathcal{U} = 0 \), which is degenerate: \( T_0(U) = c(X) \). Comp' of two reflns.

- \( G(X) \), glide refln, glide by \( \mathcal{U} \), refln across \( \ell \), \( \mathcal{U} \mathcal{U} \ell \).
  
  orbit reversing, not involvin, no fixed pts (unless \( \mathcal{U} = 0 \)) comp' of three reflns.