This exam contains 7 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you are applying a theorem, you should indicate this fact**, and explain why the theorem may be applied.
- **Do not trivialize a problem.** If you are asked to prove a theorem, you cannot just cite that theorem.
- **Organize your work** in a reasonable, tidy, and coherent way. Work that is disorganized and jumbled that lacks clear reasoning will receive little or no credit.
- **Unsupported answers will not receive full credit.** An answer must be supported by calculations, explanation, and/or algebraic work to receive full credit. Partial credit may be given to well-argued incorrect answers as well.
- If you need more space, use the back of the pages. Clearly indicate when you have done this.

Do not write in the table to the right.

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You may use the following matrices and computations on the exam without defining or proving them.

\[
R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad F_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}, \quad R_\varphi R_\theta = R_{\varphi + \theta}, \quad F_\varphi F_\theta = R_{2(\varphi - \theta)}
\]

\[
\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta), \quad \sin(2\theta) = 2\sin(\theta)\cos(\theta)
\]
1. Let \( \ell \) be the line \((5, 12) \cdot X = 17\).

(a) (4 points) Find a parametric representation of \( \ell \).

\[(5, 12) \cdot (1, 1) = 17, \text{ so } (1, 1) \text{ is on line.}
\]

\[(5, 12) \perp \ell, \text{ so } -12/5 \parallel \ell
\]

\[\ell = \left\{ (1, 1) + s(-12, 5) : s \in \mathbb{R} \right\}
\]

(Other answers possible)

(b) (4 points) Find a special normal form of \( \ell \).

Need a unit length normal vector:

\[
\frac{(5, 12)}{\sqrt{25 + 144}} = \frac{(5, 12)}{\sqrt{169}} = \left(\frac{5}{13}, \frac{12}{13}\right)
\]

So take eqn for \( \ell \), divide both sides by 13:

\[
\frac{1}{13} (5, 12) \cdot X = 17
\]

\[
\left(\frac{5}{13}, \frac{12}{13}\right) \cdot X = \frac{17}{13}
\]

(Other answers possible, e.g. 

\[
\left(\frac{5}{13}, \frac{12}{13}\right) \cdot (X - (1, 1)) = 0.
\]

2. (6 points) Prove that \( \overline{PQ} = \{aP + bQ \mid a + b = 1, a, b \geq 0\} \).

We know

\[
\overline{PQ} = \left\{ P + s(A-P) \mid 0 \leq s \leq 1 \right\}
\]

\[
= \left\{ (1-s)P + sQ \mid 0 \leq s \leq 1 \right\}
\]

\[
= \{aP + bQ \mid a = 1-b, 0 \leq b \leq 1\} \quad b = s, a = 1-s
\]

\[
= \{aP + bQ \mid a+b=1, a, b \geq 0\} \quad \text{if } a = 1-b \text{ and } 0 \leq b \leq 1, a \in [0,1] \text{ too.}
\]
3. Recall that \( \arccos z = \int_{z}^{1} \frac{1}{\sqrt{1-t^2}} \, dt \) and we define \( \pi = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} \, dt. \)

(a) (6 points) Let rays \( p \) and \( q \) emanate from a common vertex, with direction indicators \((1, 3)\) and \((2, -1)\). Find \( |\angle(p, q)| \). An answer with an integral is fine.

\[
\begin{align*}
U \cdot V &= \frac{(1, 3)}{\sqrt{10}} \cdot \frac{(2, -1)}{\sqrt{5}} = \frac{-1}{\sqrt{50}} = -\frac{1}{2\sqrt{5}} \\
|\angle(p, q)| &= \int_{-\frac{1}{2\sqrt{5}}}^{1} \frac{1}{\sqrt{1-t^2}} \, dt
\end{align*}
\]

(b) (6 points) Use calculus to prove that \( \arccos 0 = \frac{\pi}{2} \).

\[
\arccos 0 = \int_{0}^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-t^2}} \, dt = \frac{1}{2} \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} \, dt = \frac{\pi}{2}
\]

(c) (6 points) Prove \( \arccos \frac{1}{\sqrt{2}} = \pi/4 \) using the methods of this course.

Let \( U = (1, 0) \), \( V = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \), \( W = (0, 1) \) be unit vectors for angles with measure \( \theta, \varphi \) as shown. Then \( \theta = \varphi \), since

\[
\theta = \arccos(U \cdot V) = \arccos(\frac{1}{\sqrt{2}}) \quad \{\text{same} \}
\]

We also see \( \theta + \varphi = \arccos(U \cdot W) = \arccos(0) = \frac{\pi}{2} \).

Thus \( \theta + \varphi = 2\theta = \frac{\pi}{2} \)

\( \Rightarrow \quad \theta = \arccos(\frac{1}{\sqrt{2}}) = \frac{\pi}{4} \).
4. (8 points) Prove $|\angle ABC| = |\angle DBE|$ using the definitions and methods of this course.

Let $U = \frac{A - B}{||A - B||}$, $V = \frac{C - B}{||C - B||}$. Then $U, V$ are unit DI’s for $|\angle ABC|$, and $-U, -V$ are unit DI’s for $|\angle DBE|$. Thus

$$|\angle ABC| = \arccos(U \cdot V)$$

$$|\angle DBE| = \arccos((-U) \cdot (-V))$$

$$= \arccos(U \cdot V)$$

$$= |\angle ABC|,$$

as desired.

5. (5 points) Given two vectors $U$ and $V$, prove $||U + V||^2 = ||U||^2 + ||V||^2 + 2U \cdot V$ using methods from class.

$$||U + V||^2 = (U + V) \cdot (U + V)$$

$$= U \cdot U + U \cdot V + V \cdot U + V \cdot V$$

$$= ||U||^2 + ||V||^2 + 2U \cdot V$$
6. (a) (4 points) Complete the definition: $\mathcal{U}(X)$ is an isometry of $\mathbb{R}^2$ if...

\[ \| \mathcal{U}(p) - \mathcal{U}(q) \| = \| p - q \| \quad \forall \ p, q \in \mathbb{R}^2 \]

(b) (6 points) Let $\mathcal{U}$ and $\mathcal{V}$ be isometries of $\mathbb{R}^2$. Prove that the composition $\mathcal{U} \circ \mathcal{V}$ is an isometry.

\[ \| \mathcal{U}(\mathcal{V}(p)) - \mathcal{U}(\mathcal{V}(q)) \| = \| \mathcal{V}(p) - \mathcal{V}(q) \| \]

\[ = \| p - q \| \]

(b/c) $\mathcal{U}$ is an isometry

(b/c) $\mathcal{V}$ is an isometry

(c) (12 points) Let $\ell = \{(1, 2) + t(2, 3)\}$, where $t \in \mathbb{R}$. Find the matrix formula $\mathcal{M}_\ell(X)$ for the reflection across the line $\ell$. Your answer should include exact values in the matrix, not trig functions, but you do not need to multiply out your entire formula.

\[ \mathcal{M}_\ell(X) = F_\theta (X - p) + p \]

\[ = \begin{bmatrix} -\frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{bmatrix} \left( X - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]

\[ \cos \theta = \frac{4}{13} - \frac{9}{13} = -\frac{5}{13} \]

\[ \sin \theta = 2 \cdot \frac{3 \cdot 2}{13} = \frac{12}{13} \]
7. (10 points) Find the matrix formula $\mathcal{R}(X)$ for the rotation by $2\pi/3 = 120^\circ$ centered at the point $(-3,4)$. Your answer should include exact values in the matrix, not trig functions, but you do not need to multiply out your entire formula.

\[
\mathcal{R}(X) = M_{2\pi/3} = \begin{bmatrix} \cos \frac{2\pi}{3} & -\sin \frac{2\pi}{3} \\ \sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}
\]

\[
\mathcal{R}(X) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} (X - \begin{bmatrix} -3 \\ 4 \end{bmatrix}) + \begin{bmatrix} -3 \\ 4 \end{bmatrix}
\]

8. Suppose lines $l$ and $m$ intersect at $C$, with angles $\alpha$ and $\beta$ as shown below.

(a) (4 points) Write down matrix formulas for $M_l(X)$ and $M_m(X)$, the reflections across these lines. Make sure to define and label (on the diagram) any new letters or notation you introduce, such as $\theta$.

$b) (7 points)$ Now consider the composition $M_m \circ M_l(X) = M_m(M_l(X))$. Use the methods of this course (and your answers to the previous part) to identify the resulting isometry. If it is a translation, what is the translation vector? If it is a rotation, by how much, and centered at which point? If it is a reflection, what is the mirror? Citing the answer from memory will result in a few points; for full credit you must justify your answer.

\[
M_m \circ M_l(X) = M_m \left( F_{\alpha}(X-C) + C \right)
\]

\[
= F_{\theta + \beta} \left( \left[ F_{\alpha}(X-C) + C \right] - C \right) + C
\]

\[
= R_{2\pi} \left( \left[ F_{\alpha}(X-C) + C \right] - C \right)
\]

\[
= R_{2\beta} \left( X-C \right) + C
\]

i.e. rotation by $2\beta$, centered at $C$. 

9. (12 points) Indicate whether each statement is True or False by circling the appropriate answer. Justify your answer with definitions, theorems and methods from this course. (If false, be specific with your explanation; e.g. tell me what part of a definition is not satisfied, or give an example to show the statement is false, etc.)

(a) For any \( A, B \) and \( C \), the barycentric coordinates of the origin \((0,0)\) are always \((0,0,0)\).  

\[
\begin{align*}
(0,0,0) & \not\text{ valid BC's, because } 0+0+0 \neq 1. \\
& \text{True} \quad \text{False}
\end{align*}
\]

(b) If \( U(X) \) is an isometry, then \( U(aP + bQ) = aU(P) + bU(Q) \) for all \( a, b \in \mathbb{R} \) and \( P, Q \in \mathbb{R}^2 \).

\[
\begin{align*}
\text{Only valid in general if } a+b=1. \\
& \text{True} \quad \text{False}
\end{align*}
\]

Counter-example: \( P = (0,0), \ Q = (0,1), \ U(X) = X + (1,0) \), \( a=b=1 \)

\[
U(aP+bQ) = U(0,1) = (1,1)
\]

\[
\|U(0,0)\| + \|U(0,1)\| = (1,0) + (1,1) = (2,1)
\]

(c) In a glide reflection, the “glide” is always perpendicular to the mirror line.

The glide is \( \parallel \) mirror, not \( \perp \).