This exam contains 7 pages (including this cover page) and 11 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- If you are applying a theorem, you should indicate this fact, and explain why the theorem may be applied.

- Do not trivialize a problem. If you are asked to prove a theorem, you cannot just cite that theorem.

- Organize your work in a reasonable, tidy, and coherent way. Work that is disorganized and jumbled that lacks clear reasoning will receive little or no credit.

- Unsupported answers will not receive full credit. An answer must be supported by calculations, explanation, and/or algebraic work to receive full credit. Partial credit may be given to well-argued incorrect answers as well.

- If you need more space, use the back of the pages. Clearly indicate when you have done this.

Do not write in the table to the right.

You may use the following results on the exam without defining or proving them.

The distance between points \((a, b)\) and \((a, d)\) in the Poincaré Half Plane is \(|\ln(d/b)|\).

The distance between points \(P_1\) and \(P_2\) on the line \((x - \omega)^2 + y^2 = \rho^2\), with angles \(t_1 = |\angle(\omega + \rho, 0)(\omega, 0)P_1|\) and \(t_2 = |\angle(\omega + \rho, 0)(\omega, 0)P_2|\) is

\[
\ln \left[ \frac{\csc t_2 - \cot t_2}{\csc t_1 - \cot t_1} \right]
\]
1. Let \( ABCD \) be quadrilateral which is simple, but not necessarily a parallelogram, trapezoid, or other familiar shape. Let \( W, X, Y \) and \( Z \) be the midpoints of the sides, as shown in the generic diagram below.

(a) (4 points) What is \( W \) in terms of \( A \) and \( B \)? Write similar expressions for \( X, Y, \) and \( Z \).

\[
W = \frac{A + B}{2} \quad X = \frac{B + C}{2} \quad Y = \frac{C + D}{2} \quad Z = \frac{A + D}{2}
\]

(b) (8 points) Prove that \( WXYZ \) is a parallelogram.

\[
W + Y = \frac{A + B + C + D}{2} = \frac{B + C + A + D}{2} = X + Z.
\]

Many methods possible, e.g.:

(Could show any of the other conditions in Thm 8.4)

2. (5 points) Find the area of quadrilateral \( ABCD \) if \( A = (-1, 0), B = (5, 3), C = (6, 7) \) and \( D = (0, 4) \). If you use a formula, explain why it applies. (You can use the grid if it’s helpful, or ignore it.)

\[
B - A = (5, 3) - (-1, 0) = (6, 3) = C - D
\]

\[
D - A = (1, 4) = C - B
\]

Thus \( ABCD \) is a \( \text{||g} \)ram and its area is

\[
\left| \det \begin{bmatrix} u & v \\ x & y \end{bmatrix} \right| = \left| \det \begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix} \right| = \left| 42 - 3 \right| = 21
\]
3. (5 points) Prove that the perpendicular bisectors of (the sides of) \(\triangle ABC\) all intersect in a point \(J\) which is equidistant to all three vertices. (You can use the diagram below for convenience but your argument must apply to all triangles.)

Let \(J\) be the intersection of \(\perp\) bisectors of \(\overline{AB}\) and \(\overline{BC}\). (Because \(\overline{AB}\), \(\overline{BC}\) not \(\parallel\), the \(\perp\) bisectors aren’t \(\parallel\) either, hence intersect.)

Thus \(|\overline{JA}| = |\overline{JB}|\) and \(|\overline{JB}| = |\overline{JC}|\). By transitivity, \(|\overline{JA}| = |\overline{JC}|\). Hence \(J\) is also on \(\perp\) bisector of \(\overline{AC}\).

4. A corollary to Theorem 7.16 in the book says that, unless \(\triangle ABC\) is equilateral, there is a unique line through its centroid, orthocenter and circumcenter. This line is known as its Euler Line.

(a) (7 points) Let \(\triangle ABC\) be isosceles, with \(\overline{AC} \cong \overline{BC}\), as shown below. Prove the Euler Line of \(\triangle ABC\) is the median from \(C\), which intersects \(\overline{AB}\) at \(D\) as shown.

D midpt of \(\overline{AB} \Rightarrow \overline{AD} \cong \overline{BD}\). By SSS congruence, \(\triangle ADC \cong \triangle BDC\). Thus

(1) \(\angle ADC\) and \(\angle BDC\) are congruent and combine to form a right angle, so each measures \(\pi/2 = 90^\circ\). Thus \(\overrightarrow{CD}\) is an altitude...

(2) ... and, because \(\overline{AD} = \overline{BD}\), \(\overrightarrow{CD}\) is \(\perp\) bisector, too.

Thus \(\overrightarrow{CD}\) is a median, \(\perp\) bisector and altitude, so contains the centroid, circumcenter and orthocenter; thus it’s the Euler line.

(b) (3 points) Why does an equilateral \(\triangle ABC\) not have an Euler Line?

In an equilateral \(\triangle ABC\), \(G = H = J\), so those “three” points don’t determine a line.
5. (5 points) Let \( A = (0, 0), B = (5, 2) \) and \( C = (1, 7) \). Find the centroid of \( \triangle ABC \), in both barycentric and rectangular coordinates.

\[
\left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)_{\triangle ABC} = (2, 3)
\]

6. In \( \triangle ABC \) below, \( \overline{AE} \) and \( \overline{CD} \) are medians, intersecting at the centroid \( G \). Let \( x = ||\triangle ABC|| \), the area of the large triangle.

(a) (5 points) Explain why \( ||\triangle ACE|| = \frac{1}{2}x \).

(b) (3 points) Find \( ||\triangle ACG|| \) in terms of \( x \). Justify your answer.

Because \( G \) is the centroid, it’s \( \frac{1}{3} \) of way from \( E \) to \( A \), so \( \overline{GE} = \frac{1}{3}\overline{GA} \).

Thus (treating \( \overline{AE} \) and \( \overline{GE} \) as bases, and an altitude from \( C \)),

\[
||\triangle ACG|| = \frac{2}{3} ||\triangle ACE|| = \frac{2}{3} \cdot \frac{1}{2} x = \frac{1}{3} x
\]

(or can use all 3 medians to split \( \triangle \) into 6 \( \triangle \)s of equal area \( \frac{x}{6} \), as in HW, and combine two of them to form \( \triangle ACG \) with area \( \frac{2x}{6} = \frac{1}{3} x \)).
7. (a) (5 points) Let \( m \) be a circle centered at \( C \) with radius \( \rho \). Prove: if \( X \in m \), then \( X' = X \), where \( X' \) is the reflection (or inversion) of \( X \) across \( m \).

\[ X' \in \overrightarrow{Ox} \text{ chosen such that } |\overrightarrow{OX} \cdot \overrightarrow{OX}'| = \rho^2 \]

\[ \rho |\overrightarrow{OX}'| = \rho^2 \]

\[ |\overrightarrow{OX}'| = \rho \]

The only possibility is \( X' = X \).

(without \( \mathbb{R} \), we only know \( X' \in m \), not necessarily \( X' = X \).

(b) (5 points) Let \( m \) be a circle centered at \( C \), and suppose \( d \) is a circle containing \( C \) which intersects \( m \) at points \( P \) and \( Q \) as shown. Sketch \( d' \), the reflection of \( d \) across \( m \). Briefly justify your answer.

\[ C \in d \Rightarrow \overrightarrow{CO} \in d' \Rightarrow d' \text{ is a line.} \]

\[ P, Q \in m \text{ are fixed, so } P, Q \in d' \]

Thus \( d' = \overrightarrow{PQ} \)

(c) (5 points) Let \( m \) be the circle \( x^2 + y^2 = 1 \) and \( c \) the circle \( x^2 + y^2 = 4 \). Sketch and find an equation (with brief justification) for \( c' \), the reflection of \( c \) across \( m \).

\[ X = (2, 0) \in c \text{ sent to } X' \in \overrightarrow{Ox} (= \text{pos. x-axis}) \text{ s.t.} \]

\[ \|X\| \cdot |X'| = 1 \Rightarrow \|X'\| = \frac{1}{\sqrt{2}} \Rightarrow X' = \left( \frac{1}{\sqrt{2}}, 0 \right) \]

Similarly, \((0, \pm 2) \mapsto (0, \pm \frac{2}{\sqrt{2}}), (-2, 0) \mapsto (-\frac{2}{\sqrt{2}}, 0) \)

Thus \( c' : x^2 + y^2 = \left( \frac{2}{\sqrt{2}} \right)^2 \)

(d) (5 points) Let \( m \) be the circle \( x^2 + y^2 = 1 \) and \( \ell \) the line \( y = -x \). Sketch and find an equation (with brief justification) for \( \ell' \), the reflection of \( \ell \) across \( m \).

\[ \ell \Rightarrow \ell \Rightarrow \ell' \text{ (so } \ell' \text{ is a line)} \]

\[ \ell \Rightarrow \ell' \]

\[ P, Q \in \ell \text{ are on } m, \text{ so } P, Q \in \ell' \]

Thus \( \ell' \) is also the line \( y = x \)
8. Let $A = (-3, 1)$, $B = (-1, 1)$ and $C = (3, 1)$.

(a) (9 points) Sketch the (Poincaré) lines $AB$ and $BC$. Find the equation for each line.

(b) (5 points) Find the length of segment $BC$. (For full credit, your final answer shouldn’t contain any trig functions, but you don’t have to evaluate any logarithms.)

9. Let $\ell$ be the line directed by $A = (-2, 0)$ and $B = (\infty, 0)$.

(a) (5 points) Sketch $\ell$ on the grid below. Then sketch and give an equation for a line $m$ which contains the point $(0, 2)$ and is asymptotically parallel to $\ell$.

(b) (3 points) Give an equation for a line $k$ containing $(1, 1)$ which is ultra parallel to $\ell$, or explain why none exists.
10. (5 points) Given \( \triangle ABC \), let \( D \) and \( E \) be the midpoints of \( AC \) and \( BC \) as shown. Using any appropriate methods from the course, prove \( DE \parallel AB \) and \( |DE| = \frac{1}{2}|AB| \).

Method 1
\[
E - D = \frac{B + C}{2} - \frac{A + C}{2} = \frac{1}{2}(B - A)
\]
Thus \( E - D \parallel B - A \) and \( E - D \) is half as long as \( B - A \).

Method 2
Because \( D, E \) are midpoints of \( AC, BC \), we have \( \frac{|AC|}{|EC|} = \frac{|EC|}{|BC|} = 2 \).
Note also that \( \triangle ACB \) and \( \triangle BCE \) share the angle \( \gamma \). Thus \( \triangle ACB \sim \triangle BCE \) by SAS

Similarity (\#2 on previous page). That means
\[
(a) \quad \frac{|AB|}{|DE|} \text{ is also } 2, \text{ as needed, and}
\]
\[
(b) \quad \angle CEB = \angle CAB \Rightarrow \overline{DE} \parallel \overline{AB}
\]
(corresponding angles)

11. For each of the following statements, indicate whether it is **True** or **False** by circling the corresponding answer. Briefly justify your answer.

(a) (4 points) Let \( ABCD \) be a simple quadrilateral. If \( A + B = C + D \), then \( ABCD \) is a parallelogram.

**True**

(b) (4 points) If \( \triangle DEF \) is the image of \( \triangle ABC \) under a conformal affinity, then their areas must be equal.

**True**

Conformal affinities can scale \( \mathbb{R}^2 \), changing the area
(Think: similarity)