The following is a not-necessarily comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I’ll update things as soon as possible.

3.8: Let $P = (3, 2)$, $Q = (2, 0)$ and $R = (4, 5)$. Here are the steps to measure the angle $\angle PQR$. You might not have shown this much detail, which is fine, but it’s vital that you find unit direction indicators for the angle.

1. Find a unit vector $U$ which is a direction indicator for $\overrightarrow{QP}$:
   $$U = \frac{P - Q}{||P - Q||} = \frac{(1, 2)}{\sqrt{1 + 4}} = \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$$

2. Find a unit vector $V$ which is a direction indicator for $\overrightarrow{QR}$:
   $$V = \frac{R - Q}{||R - Q||} = \frac{(2, 5)}{\sqrt{4 + 25}} = \left( \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right)$$

3. Find the dot product of $U$ and $V$:
   $$U \cdot V = \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \cdot \left( \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right)$$
   $$= \left( \frac{2}{\sqrt{145}} + \frac{10}{\sqrt{145}} \right)$$
   $$= \frac{12}{\sqrt{145}}$$

4. Find the measure of $\angle PQR$:
   $$|\angle PQR| = \int_{U \cdot V}^{1} \frac{dt}{\sqrt{1 - t^2}} = \int_{\frac{12}{\sqrt{145}}}^{1} \frac{dt}{\sqrt{1 - t^2}} = 0$$
   (Wow, that’s ugly!)

3.15: The situation looks something like this:

We can assume Proposition 3.19, but have to be careful about only applying it to two angles at once, each of which must meet along a common ray in the interior of the outer rays. Then

$$|\angle (p, s)| = |\angle (p, q)| + (|\angle (q, s)|)$$
$$= |\angle (p, q)| + (|\angle (q, r)| + |\angle (r, s)|)$$
3.16: There are many possible answers. Here’s one. Let $p$ be the positive $x$-axis, $q$ the positive $y$-axis, $r$ the negative $x$-axis and $s$ the negative $y$-axis:

With our definitions, $|\angle(p, s)| = \pi/2$, but:

$$|\angle(p, q)| + |\angle(q, r)| + |\angle(r, s)| = \pi/2 + \pi + \pi = 3\pi/2.$$  

3.22: We did this problem as group work in class, although it was phrased a little bit differently; on that sheet I gave you vectors whose dot product was $1/2$ and essentially asked you to find $\arccos(U \cdot V)$. This problem asks for $\arccos(1/2)$, leaving you to find the vectors. Since this problem is important, I’ll include a longer-than-normal discussion here.

I assigned this problem because it helps demonstrate a quirk of our definition of angle measure. Go back and look at the solution for 3.8 to refresh your memory: to find the measure of an angle, we need two unit vectors $U$ and $V$ which define the angle, and then the measure is defined as an integral:

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}}$$

That definition is great mathematically, but it’s a problem from a practical point of view since we can’t actually evaluate that integral! So we have to resort to some indirect reasoning:

1. We define $\pi$ to be the measure of an angle for which $U \cdot V = -1$. As it happens, any such $U$ and $V$ represent a straight angle, as we would expect. [You actually proved on the first homework assignment that $U = -V$ in this case.]

2. In class, we asked what would happen if $U \cdot V = 0$. By comparing the resulting integral to the definition of $\pi$, we decided such an angle would have measure $\pi/2$. So that’s why a right angle has measure $\pi/2$.

3. On the worksheet (or Example 3.27 in the book), we chose $U$, $V$ and $W$ which happen to construct two angles which must combine to form one right angle. (Draw a picture of these vectors!) We know those angles have equal measure, since $U \cdot V = V \cdot W$, and the measure of an angle is entirely determined by the dot product of its (unit) direction indicators. So if two equal angles add to $\pi/2$, they must each measure $\pi/4$.

Following this pattern, you might try to pick $U$, $V$ and $W$ which form two angles which add up to $\pi/2$ such that $|\angle UOV|$ is twice as large as $|\angle VOW|$. Then one would have to be $\pi/3$ and the other $\pi/6$. Using our prior knowledge of precalculus and 30-60-90 triangles, you can even say what $V$ would be:

$$U = (1, 0)$$
$$V = (1/2, \sqrt{3}/2)$$
$$W = (0, 1)$$

The problem is that its very hard to show that one of those angles is twice the other, i.e.:
\[
\int_{U \cdot V}^1 \frac{dt}{\sqrt{1 - t^2}} = 2 \int_{V \cdot W}^1 \frac{dt}{\sqrt{1 - t^2}} \\
\int_{1/2}^1 \frac{dt}{\sqrt{1 - t^2}} = 2 \int_{\sqrt{3}/2}^1 \frac{dt}{\sqrt{1 - t^2}}
\]

It turns out to be easier to choose:

\[
U = (1, 0) \\
V = (1/2, \sqrt{3}/2) \\
W = (-1/2, \sqrt{3}/2) \\
X = (-1, 0)
\]

You can verify that \(U \cdot V, V \cdot W\) and \(W \cdot X\) are all equal, so the angles have equal measure:

\[
\int_{U \cdot V}^1 \frac{dt}{\sqrt{1 - t^2}} = \int_{V \cdot W}^1 \frac{dt}{\sqrt{1 - t^2}} = \int_{W \cdot X}^1 \frac{dt}{\sqrt{1 - t^2}}
\]

All three of them must add up to \(\pi\) by our previous work. (Problem 3.15!) Hence each is one third of \(\pi\) – also known as \(\pi/3\). Then you can combine angles.

\[
|\angle UOV| = \arccos 1/2 = \pi/3
\]

\[
|\angle UOW| = \pi/3 + \pi/3 = 2\pi/3
\]

\[
|\angle UOV| = \arccos \left(-\frac{1}{2}\right) = 2\pi/3.
\]

4.1: Let’s give our function a name: define \(U(x_1, x_2) = (x_1, -x_2)\). We want to check that \(U\) is an isometry.

Many people used the good-old distance formula to show \(\|U(P) - U(Q)\| = \|P - Q\|\). I’ll do that here, but square the distances to avoid the square roots:

\[
\|U(P) - U(Q)\|^2 = \|U(p_1, p_2) - U(q_1, q_2)\|^2 \\
= \|(p_1, -p_2) - (q_1, -q_2)\|^2 \\
= \|(p_1 - q_1, -p_2 + q_2)\|^2 \\
= (p_1 - q_1)^2 + (-p_2 + q_2)^2
\]

Meanwhile,

\[
\|P - Q\|^2 = \|(p_1, p_2) - (q_1, q_2)\|^2 \\
= (p_1 - q_1)^2 + (p_2 - q_2)^2
\]

The result follows from the fact that \((-p_2 + q_2)^2 = (p_2 - q_2)^2\).

4.10: When I assigned this problem, we had proven that a translation, \(T_V(X) = X + V\) is an isometry. Nearly any translation will work for this problem – we just need to avoid \(V = 0\). If \(U(X) = X + V\), then

\[
U(aP + bQ) = aP + bQ + V \\
aU(P) + bU(Q) = aP + V + bQ + V = aP + bQ + 2V
\]
You could construct a specific example by choosing values like $a = b = 1$, $P = (0, 0)$, $Q = (0, 1)$ and $V = (0, 1)$.

A: Let $U$ and $V$ be isometries, so $||U(X) - U(Y)|| = ||X - Y||$ and $||V(X) - V(Y)|| = ||X - Y||$ for any points $X$ and $Y$. Then:

\[
||U(V(P)) - U(V(Q))|| = ||V(P) - V(Q)||
\]

(because $U$ is an isometry)

\[
= ||P - Q||
\]

(because $V$ is an isometry)

Thus $U \circ V(X) = U(V(X))$ is an isometry.

B: Assuming $a + b \neq 0$, we can rewrite $aP + bQ$ as

\[
aP + bQ = (a + b) \left[ \frac{a}{a+b}P + \frac{b}{a+b}Q \right].
\]

Put differently, this means $aP + bQ + cR = (a + b)S + cR$, for $S = \frac{a}{a+b}P + \frac{b}{a+b}Q$. By Lemma 4.4,

\[
U((a + b)S + cR) = (a + b)U(S) + cU(R)
\]

\[
= (a + b)U \left( \frac{a}{a+b}P + \frac{b}{a+b}Q \right) + cU(R)
\]

Now notice that $\frac{a}{a+b} + \frac{b}{a+b} = 1$, so we can use Lemma 4.4 again:

\[
U((a + b)S + cR) = (a + b)U \left( \frac{a}{a+b}P + \frac{b}{a+b}Q \right) + cU(R)
\]

\[
= (a + b) \left[ \frac{a}{a+b}U(P) + \frac{b}{a+b}U(Q) \right] + cU(R)
\]

\[
= aU(P) + bU(Q) + cU(R)
\]

This all assumed $a + b = 0$. What if $a + b = 0$, so that we can’t divide by $(a + b)$? Then $c = 1$ (remember, $a + b + c = 1$!), and although it’s possible that $a$ and $b$ could be 1 and $-1$, they can’t both be $-1$. So either $a + c = \neq 0$ or $b + c \neq 0$. If $a + c \neq 0$, we can go do the same work using $(a + c)$, $P$ and $R$ instead of $(a + b)$, $P$ and $Q$. Similarly, if $b + c \neq 0$ we can use $(b + c)$, $Q$ and $R$. 