The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I’ll update things as soon as possible.

7.4: The triangles for which the orthocenter is also a vertex are right triangles. As described on the assignment handout, for full credit you should explain both directions: if a triangle is a right triangle, then its orthocenter is at a vertex, and if a triangle’s orthocenter is at a vertex, then it’s a right triangle.

Start with a right triangle $\triangle ABC$ with right angle at $C$. Then $\overrightarrow{AC}$ is the altitude from $A$, and $\overrightarrow{BC}$ is the altitude from $B$. (Draw a picture and convince yourself of this!) Since two altitudes intersect at $C$, all three will intersect there, and $C$ is the orthocenter.

Conversely, suppose one of the vertices – let’s say $C$ – is the orthocenter of a triangle. Then the altitude from $A$ goes through $C$, which means the altitude actually is $\overrightarrow{AC}$, and by definition of altitude, $\overrightarrow{AC} \perp \overrightarrow{BC}$.

If two sides of $\triangle ABC$ are perpendicular, then $\triangle ABC$ is a right triangle.

7.6: Following the hint on the assignment and in class, let’s use barycentric coordinates. If $\triangle ABC$ is equilateral, then $a = b = c$, so the incenter (by Proposition 7.19) is

$$\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right) \triangle = \left(\frac{a}{a+a+a}, \frac{a}{a+a+a}, \frac{a}{a+a+a}\right) \triangle = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \triangle,$$

which is the centroid. Conversely, suppose the centroid and incenter are equal:

$$\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right) \triangle = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \triangle.$$

Equating the parts of the barycentric coordinates (and multiplying by 3) gives three equations:

$$3a = a + b + c$$
$$3b = a + b + c$$
$$3c = a + b + c$$

You can check that this system of equations is satisfied if and only if $a = b = c$, so that the triangle is equilateral. (The fastest is to note that $3a = 3b = 3c$.)

7.13: On the assignment handout, I said you should show more work than a GeoGebra picture for this problem, but don’t have to write out systems of equations and their solutions—once you demonstrate that you know the equations of perpendicular bisectors, altitudes, or whatever, then you can use a calculator, wolfram alpha, or another tool (even GeoGebra!) to find their intersections. I’ve included a GeoGebra picture below for you to check your answers. I’ve labeled the vertices of the triangle $A$, $B$ and $C$. The blue dashed lines are the altitudes, with feet $D$, $E$ and $F$ and orthocenter $H$. The red dotted lines are the perpendicular bisectors with circumcenter $J$. The circumradius is the distance from $J$ to any of the vertices. (They’re all the same distance from $J$; that’s the whole point!)

$$\|J - A\| = \sqrt{(5/2 - 0)^2 + (13/2 - 0)^2} = \sqrt{97/2}$$
The Euler line is the dark black line through $H$, $J$ and the centroid $G$. You can use techniques from Chapter 1 to write down an equation of it – for example,

$$aH + bG = a(-4, 9) + b(1/3, -4/3), \quad a + b = 1$$

7.20: It’s possible to set up huge systems of equations with the barycentric coordinates of the incenter and orthocenter for this problem, and show they’re equal if and only if $a = b = c$, but that would be... messy. A geometric approach seems simplest to me for this problem. If you had trouble solving this, use the diagram below (draw a fresh copy for each direction!) and fill in the details as you read through the proof.

The crucial observation, as mentioned on the assignment handout, is that the incenter and orthocenter are equal if and only if each angle bisector is also an altitude. Why is that? The angle bisector from a vertex is the unique line through the vertex and the incenter $I$, and the altitude from a vertex is the unique line through a vertex and the orthocenter $H$. Hence if $H = I$, all the angle bisectors and altitudes from each vertex are the same. Conversely, if the angle bisectors are all altitudes, the intersections $I$ and $H$ of the bisectors and altitudes must also be the same point.

So now we need to prove the three angle bisectors are altitudes if and only if a triangle is equilateral.

Suppose $\triangle ABC$ is equilateral. We need to show an angle bisector is also an altitude, so draw the bisector of $\angle BAC$, and label its intersection with $BC$ as point $D$. We have $\triangle BAD \cong \triangle CAD$ by SAS: $\overline{BA} \cong \overline{CA}$, $\angle BAD \cong \angle CAD$, and both triangles share the side $\overline{AD}$. By CPCTC, angles $\angle ADB$ and $\angle ADC$ are congruent and add to $\pi$. Hence they are right angles, and $\overline{AD}$ is an altitude. We could repeat the same work for the other angle bisectors and altitudes in the triangle.
Conversely, suppose we know the angle bisectors are also altitudes, so starting with a fresh copy of our diagram above, the line $\overline{AD}$ is an angle bisector and an altitude. This time the triangles $\triangle ADB$ and $\triangle ADC$ are congruent by ASA, and CPCTC tells us $\overline{AB} \cong \overline{AC}$. If we repeated this with the angle bisector and altitude from $C$, we’d get $\overline{AC} \cong \overline{BC}$. Hence

$$\overline{AB} \cong \overline{AC} \cong \overline{BC}$$

and the triangle is equilateral.

A: (1) The crucial observation here is that $\triangle ADC$ and $\triangle BDC$ share the same altitude from $C$, and hence have the same height $h$:

![Diagram of triangle ABC with altitude h from C to D]

Because $D$ is the midpoint of $\overline{AB}$, we also know $|\overline{AD}| = |\overline{BD}|$. Thus the triangles have the same base length, which we could call $b$. Hence $\|\triangle ADC\| = \|\triangle BDC\| = \frac{1}{2}bh$.

(2) Using the theorem we just proved, we see $\|\triangle ADG\| = \|\triangle BDC\|$. It’s the same situation as in part (1), but with $G$ in place of $C$. Similarly, using $\overline{BE}$ and $\overline{CE}$ as their bases, $\|\triangle BEG\| = \|\triangle CEG\|$. Finally, $\|\triangle AFG\| = \|\triangle CFG\|$ using $\overline{AF}$ and $\overline{CF}$ as bases.

![Diagram of triangle ABC with additional points G, E, F]

(3) Using the labels I suggested in the problem, $\|\triangle ADC\| = \|\triangle BDC\|$ implies

$$2z + x = 2y + x.$$

Thus $2z = 2y$ and $z = y$. Using the same argument with $\|\triangle BEA\| = \|\triangle CEA\|$, we find that $2z = 2x$ and $z = x$. Overall, we therefore have $x = y = z$. 

Jonathan Rogness <rogness@math.umn.edu>  
November 15, 2017