The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I’ll update things as soon as possible.

**A:** As mentioned in the hints, we did this problem in class, except the composition was in the other order. See page 13 of the class notes for Chapter 6. Using similar notation, with \( Q = P - \frac{1}{2} U \), we have the following diagram:

![Diagram](image)

Assuming \( \ell \) and \( k \) form an angle of \( \theta \) with the horizontal, our goal is to show

\[
M_\ell \circ T_U(X) = M_\ell(X + U) = F_\theta \left((X + U) - P\right) + P
\]

is the same as

\[
M_k(X) = F_\theta(X - Q) + Q = F_\theta \left(X - \left(P - \frac{1}{2} U\right)\right) + \left(P - \frac{1}{2} U\right).
\]

There are a few ways to write this out, but in the end it amounts to splitting \( U \) into \( \frac{1}{2} U + \frac{1}{2} U \). We leave one of those pieces inside the parentheses, and move the other one out. Then it’s important to remember that \( F_\theta \left(\frac{1}{2} U\right) = -\frac{1}{2} U \).

\[
M_\ell \circ T_U(X) = F_\theta \left((X + U) - P\right) + P
= F_\theta \left(x + \frac{1}{2} U + \frac{1}{2} U - P\right) + P
= F_\theta \left(x + \frac{1}{2} U - P\right) + P + F_\theta \left(\frac{1}{2} U\right)
= F_\theta \left(x - \left(P - \frac{1}{2} U\right)\right) + P - \frac{1}{2} U
= F_\theta \left(x - Q\right) + Q
\]

**7.4:** The triangles for which the orthocenter is also a vertex are *right* triangles. As described on the assignment handout, for full credit you should explain both directions: if a triangle is a right triangle, then its orthocenter is at a vertex, and if a triangle’s orthocenter is at a vertex, then it’s a right triangle.
Start with a right triangle $\triangle ABC$ with right angle at $C$. Then $\overrightarrow{AC}$ is the altitude from $A$, and $\overrightarrow{BC}$ is the altitude from $B$. (Draw a picture and convince yourself of this!) Since two altitudes intersect at $C$, all three will intersect there, and $C$ is the orthocenter.

Conversely, suppose one of the vertices – let’s say $C$ – is the orthocenter of a triangle. Then the altitude from $A$ goes through $C$, which means the altitude actually is $\overrightarrow{AC}$, and by definition of altitude, $\overrightarrow{AC} \perp \overrightarrow{BC}$.

If two sides of $\triangle ABC$ are perpendicular, then $\triangle ABC$ is a right triangle.

**7.6:** Following the hint on the assignment and in class, let’s use barycentric coordinates. If $\triangle ABC$ is equilateral, then $a = b = c$, so the incenter (by Proposition 7.19) is

$$\theta \left( \frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c} \right) = \left( \frac{a}{a+a+a}, \frac{a}{a+a+a}, \frac{a}{a+a+a} \right) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right),$$

which is the centroid. Conversely, suppose the centroid and incenter are equal:

$$\theta \left( \frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{a}{a+b+c} \right) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right),$$

Equating the parts of the barycentric coordinates (and multiplying by 3) gives three equations:

$$3a = a + b + c$$
$$3b = a + b + c$$
$$3c = a + b + c$$

You can check that this system of equations is satisfied if and only if $a = b = c$, so that the triangle is equilateral. (The fastest is to note that $3a = 3b = 3c$.)

**7.13:** On the assignment handout, I said you should show more work than a GeoGebra picture for this problem, but don’t have to write out systems of equations and their solutions—once you demonstrate that you know the equations of perpendicular bisectors, altitudes, or whatever, then you can use a calculator, wolfram alpha, or another tool (even GeoGebra!) to find their intersections. I’ve included a GeoGebra picture below for you to check your answers. I’ve labeled the vertices of the triangle $A$, $B$ and $C$. The blue dashed lines are the altitudes, with feet $D$, $E$ and $F$ and orthocenter $H$. The red dotted lines are the perpendicular bisectors with circumcenter $J$. The circumradius is the distance from $J$ to any of the vertices. (They’re all the same distance from $J$; that’s the whole point!)

$$||J - A|| = \sqrt{(5/2 - 0)^2 + (13/2 - 0)^2} = \sqrt{97/2}$$

The Euler line is the dark black line through $H$, $J$ and the centroid $G$. You can use techniques from Chapter 1 to write down an equation of it – for example,

$$aH + bG = a(-4, 9) + b(1/3, -4/3), \quad a + b = 1$$
A: (1) The crucial observation here is that $\triangle ADC$ and $\triangle BDC$ share the same altitude from $C$, and hence have the same height $h$.

Because $D$ is the midpoint of $AB$, we also know $|AD| = |BD|$. Thus the triangles have the same base length, which we could call $b$. Hence $\|\triangle ADC\| = \|\triangle BDC\| = \frac{1}{2}bh$.

(2) Using the theorem we just proved, we see $\|\triangle ADG\| = \|\triangle BDG\|$. It’s the same situation as in part (1), but with $G$ in place of $C$. Similarly, using $BE$ and $CE$ as their bases, $\|\triangle BEG\| = \|\triangle CEG\|$. Finally, $\|\triangle AFG\| = \|\triangle CFG\|$ using $AF$ and $CF$ as bases.
(3) Using the labels I suggested in the problem, \( \| \triangle ADC \| = \| \triangle BDC \| \) implies

\[ 2z + x = 2y + x. \]

Thus \( 2z = 2y \) and \( z = y \). Using the same argument with \( \| \triangle BEA \| = \| \triangle CEA \| \), we find that \( 2z = 2x \) and \( z = x \). Overall, we therefore have \( x = y = z \).