The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I’ll update things as soon as possible.

**8.8:** In this solution I’ll use \( A, B, C \) and \( D \) to refer to the four points in the problem. The parallelogram is spanned by the following vectors.

\[
U = A - B = (3, 2) - (-7, 4) = (10, -2)
\]

\[
V = C - B = (-2, -6) - (-7, 4) = (5, -10)
\]

Then the area of the parallelogram can be computed using the determinant of a 2x2 matrix with \( U \) and \( V \) as columns:

\[
\det \begin{bmatrix} 10 & 5 \\ -2 & -10 \end{bmatrix} = -100 - (-10) = -90
\]

The area is the absolute value of that determinant, i.e. 90. Depending on how you chose your vectors or assembled your matrix, your determinant might have been 90 to begin with, so that the absolute value wasn’t necessary.

**8.23:** This problem requires a lot of proofs. Almost everybody used geometric diagrams to analyze angles, construct congruent triangles, etc. to complete their proofs. Most people proved \((i) \iff (ii)\) and \((i) \iff (iii)\) but other approaches are possible. I’ll give sketches of proofs below, leaving you to fill in the details on the diagrams about which segments and angles are congruent, etc. Regardless of which condition from \(i\) to \(iii\) we’re assuming, we always know that \(ABCD\) in the following pictures is a parallelogram. Hence the opposite sides are always parallel and congruent, by Proposition 8.4.

\[(i) \implies (ii):\] Assume \(ABCD\) is a rectangle, so all four vertex angles are congruent. Because it’s also a parallelogram, \(AB \cong CD\) and \(AD \cong BC\). Hence SAS tells us that \(\triangle DAB \cong \triangle CBA\). In particular, \(BD \cong AC\), proving \((ii)\).

\[(ii) \implies (i):\] Using the same diagram, and assuming the diagonals are congruent, SSS tells us \(\triangle DAB \cong \triangle CBA\). Hence the angles at \(A\) and \(B\) are congruent. Similarly, \(\triangle BCD \cong \triangle ADC\) and the angles at \(D\)
and $C$ are congruent. Finally, $\triangle ABC \cong \triangle DCB$, so the angles at $B$ and $C$ are congruent. Put those all together, and all for angles are congruent; hence $ABCD$ is a rectangle.

$(i) \Rightarrow (iii)$: Starting with a rectangle, we have right angles at $A$, $B$, $C$ and $D$. Labeling the other congruences we know about, SAS tells us all four of the small right triangles in the following diagram are congruent. In particular, their hypotenuses, connecting the midpoints $P$, $Q$, $R$ and $S$ are congruent. This means $PQRS$ is a rhombus, proving $(iii)$

$(iii) \Rightarrow (i)$: In that same picture, assume $PQRS$ is a rhombus. Then all four triangles are congruent by SSS. Hence the angles at $A$, $B$, $C$ and $D$ are congruent, proving $(i)$.

8.43: Most people drew kites. Here’s another possibility.

9.3: The diagram below shows each of these points and their inversions across the circle. The point $F = (0, 0)$ is inverted to $E' = \infty$, which is not shown on the picture.

9.4: You can almost use the formula given in the book for a circle inversion, with one exception: that formula assumes the mirror is centered at the origin, whereas our circle is centered at $(-8, 13)$. So we need to first
move everything so the center is at the origin, then invert, and then move it back:

\[
I(X) = \begin{cases} 
\frac{\rho^2}{||X-C||^2} (X - C) + C, & X \neq C, \infty \\
\infty, & X = C \\
C, & X = \infty 
\end{cases}
\]

where \( C = (-8, 13) \) and \( \rho = 29 \) in our case. By my quick calculations, this yields:

\[
I(0, 0) = \left( \frac{4864}{233}, -\frac{7904}{233} \right) \\
I(12, -8) = (12, -8) \text{ (this point is on the mirror, so it stays fixed!)} \\
I(\infty) = (-8, 13) \\
I(8, -13) = \left( \frac{1500}{233}, -\frac{4875}{466} \right)
\]

9.5: I'm assuming you can make pictures of these in GeoGebra; let me know if you need help with that. As for the answers themselves:

(i) The circle of radius 2 centered at \( O \) is sent to the circle of radius 1/2 centered at \( O \).

(ii) The circle of radius 3 centered at \( O \) is sent to the circle of radius 1/3 centered at \( O \).

(iii) The circle of radius 1 centered at \( (0, -1) \) goes through the origin. The origin is sent to \( \infty \), which means the image of our circle will be a line. (Lines contain \( \infty \). Circles don't.) Furthermore, the circle of radius 1 centered at \( (0, -1) \) intersects our mirror circle at the points \( (\pm \sqrt{3}/2, -1/2) \). [Draw a picture and/or solve the system of equations to see this. Ask me if you're not sure why.] Hence the image is the line through those points, which is the horizontal line \( y = -1/2 \). You could also find this via formulas in Theorem 9.5.

(iv) The line \( x_2 = -1 \) (i.e. \( y = -1 \)) includes the point at infinity, so its inversion will include the origin. It also includes the the point \( (0, -1) \), which is on the mirror and hence is fixed by the inversion. We could examine some other points, use Theorem 9.5, or draw a diagram like the one I used to prove part (ii) of the theorem, to see this will be the circle with diameter from \( (0, 0) \) to \( (0, -1) \). That’s the circle of radius 1/2 centered at \( (0, -1/2) \).

(v) The line \( x_2 = x_1 \) (i.e. the line \( y = x \)) is sent to itself, although the circle inversion is not the identity function for points on the line; points on the line but outside the circle are sent to points on the line but inside the circle, and vice versa.

(vi) The circle of radius 5/2 centered at \( (-3, 0) \) does not include the origin, so it will be sent to a circle which does not include the origin. It includes the points \( (-11/2, 0) \) and \( (-1/2, 0) \), which are inverted to \( (-2/11, 0) \) and \( (-2, 0) \), respectively. These points turn out to be the diameter of the new circle, which is therefore centered at

\[
\frac{1}{2} \left((-2/11, 0) + (-2, 0)\right) = (-12/11, 0)
\]

and has equation

\[
(x + 12/11)^2 + y^2 = (10/11)^2
\]

A: Reflect the following curves across the circle of radius \( \sqrt{5} \) centered at \( (2, 0) \).

(1) The line \( \ell : y = 2 \) includes \( \infty \), so \( \ell' \) includes its reflection, which is \( (2, 0) \), the center of the circle. The line \( \ell \) also includes the points \( (1, 2) \) and \( (3, 2) \), which are on the mirror and hence are fixed. So \( \ell' \) includes
(2, 0), (1, 2), and (3, 2), which means it’s a circle. You can check it’s the circle centered at (2, 5/4) with radius 5/4:

\[(x - 2)^2 + (y - 5/4)^2 = 25/16\]

(2) The circle \((x - 4)^2 + y^2 = 4\) goes through the center of the mirror. Hence its reflection includes \(\infty\) and must be a line. The circle also intersects the mirror at the points

\[\left(\frac{13}{4}, \pm \frac{\sqrt{55}}{4}\right)\].

Thus the reflection is the vertical line \(x = 13/4\).

(3) The line \(x + y = 2\) goes through the center of the mirror. It is reflected to the same line.