The goal of this handout is to prove Varignon’s Theorem, which says the area of a convex quadrangle $ABCD$ is twice the area of the parallelogram with vertices at the midpoints of the side $ABCD$. Our proof will depend on the following facts about a triangle with the midpoints connected.

**Theorem.** Let $D$ and $E$ be the midpoints of $AC$ and $BC$. Then $DE \parallel AB$ and $|DE| = \frac{1}{2}|AB|$.

**Proof:** We proved this in class using SAS Similarity; solutions are posted on the course website in the folder with the various activities I’ve handed out during the semester. □

**Corollary.** The four triangles in the diagram above all have area $\frac{1}{4}||\triangle ABC||$.

**Proof:** By the theorem, the base $AB$ of $\triangle ABC$ is twice as long as the base $DE$ of $\triangle DEC$. By similarity, the height of $\triangle ABC$ is twice that of $\triangle DEC$ as well. Hence $||\triangle ABC|| = 4||\triangle DEC||$, or $||\triangle DEC|| = \frac{1}{4}||\triangle ABC||$. Now you can apply the theorem using the other sides of $\triangle ABC$ as the base, and show $||\triangle EFB|| = ||\triangle FDA|| = \frac{1}{4}||\triangle ABC||$ as well. Overall, $\triangle ABC$ is divided into 4 triangles in our diagram. We’ve shown three of them have area $\frac{1}{4}||\triangle ABC||$. Hence the remaining triangle must have the same area. □

**Prove.** The area of a convex quadrangle $ABCD$ is twice the area of the parallelogram with vertices at the midpoints of the side $ABCD$.

**Proof:** In the diagram below, $E$, $F$, $G$ and $H$ are the midpoints of the sides of $ABCD$. The dotted lines are the diagonals of $ABCD$. We’ll show $||EFGH|| = \frac{1}{2}||ABCD||$.

The area of the parallelogram $EFGH$ is the area of $ABCD$ minus the area of the shaded triangles.

$$||EFGH|| = ||ABCD|| - ||\triangle AEH|| - ||\triangle BFE|| - ||\triangle CGH|| - ||\triangle DHG||$$

By the Corollary above, each shaded triangle is $1/4$ the area of a larger triangle formed with one of the diagonals. For example, $||\triangle AEH|| = \frac{1}{4}||\triangle ABD||$. Hence:

$$||EFGH|| = ||ABCD|| - \frac{1}{4}||\triangle ABD|| - \frac{1}{4}||\triangle BCA|| - \frac{1}{4}||\triangle CDB|| - \frac{1}{4}||\triangle DAC||$$

$$= ||ABCD|| - \frac{1}{4} (||\triangle ABD|| + ||\triangle CDB||) - \frac{1}{4} (||\triangle BCA|| + ||\triangle DAC||)$$

In the last line I did a bit of rearranging to match certain triangles together, because they combine to form the whole quadrilateral. Notice that $||\triangle ABD|| + ||\triangle CDB|| = ||ABCD||$. Similarly, $||\triangle BCA|| + ||\triangle DAC|| = ||ABCD||$. Thus,

$$||EFGH|| = ||ABCD|| - \frac{1}{4}||ABCD|| - \frac{1}{4}||ABCD|| = \frac{1}{2}||ABCD||.$$  □