UMTYMP Calculus III Quiz 2 review problems

1. Consider the curve \( C = C_1 \cup C_2 \cup C_3 \), where:
   - \( C_1 \) goes from the origin to \((0,3,0)\) along a straight line segment.
   - \( C_2 \) goes from \((0,3,0)\) to \((0,3,-5)\) along a straight line segment.
   - \( C_3 \) goes from \((0,3,-5)\) to \((2,3,-5)\) along a straight line segment.
   a) Sketch \( C_1, C_2 \) and \( C_3 \) in \( \mathbb{R}^3 \).
   b) Without doing any computations other than addition and multiplication of real numbers, evaluate \( \int_C yz \, dx + x \, dy + y \, dz \).

2. Explain geometrically why each of the following line integrals evaluates to zero:
   a) \( \int_C e^{\arctan x} y \cos(2y) \, ds \), where \( C \) goes from \((10,15)\) to \((10,-15)\) along a straight line segment.
   b) \( \oint_C \frac{x}{x^2+y^3+1} \, dx + \frac{y}{x^2+y^3+1} \, dy \), where \( C \) is the unit circle oriented counterclockwise.
   c) \( \int_C e^{xy} \, dx + e^{xy} \, dy \), where \( C \) is any line segment that lies on a line passing through the origin.

3. Identify the following sets as connected or disconnected, and as open or not open.
   a) \( S = \{ (x,y) \mid x \geq 0, y \geq 0, x+y \leq 1 \} \)
   b) \( T = \{ (x,y) \mid x < 0 \text{ or } x > 3 \} \)
   c) \( K = \{ (x,y) \mid |x| \geq 1 \text{ or } |y| \geq 1 \} \)
   d) \( U = \{ (x,y) \mid x^2+y^2 < 1 \text{ or } (x-3)^2+y^2 < 4 \} \)
   e) \( B = \{ (x,y) \mid x^2+y^2 < 81 \} \cup \{ (x,y) \mid x^2+y^2 > 81 \} \)
   f) \( B = \{ (x,y) \mid 0 < (x-3)^2+(y+1)^2 < 16 \} \)

4. Which of the connected sets in Exercise 3 are simply-connected?

5. Let \( \mathbf{F} = P(x) \mathbf{i} + Q(y) \mathbf{j} \) be a vector field in an open simply-connected region \( D \), and \( P \) and \( Q \) have continuous first-order partial derivatives. Is \( \mathbf{F} \) conservative? If so, find \( f \) such that \( \mathbf{F} = \nabla f \).
6. Compute $\oint_C \left( (\cos x + 1)^2 + 2y \right) dx + \left( (\sin y - 1)^2 - 5x[1 + \frac{1}{x}] \right) dy$, where $C$ is consists of the line segments from $(1, 0)$ to $(0, 1)$, from $(0, 1)$ to $(-1, 1)$, from $(-1, 1)$ to $(-1, 0)$, and the lower half of the unit circle from $(-1, 0)$ back to $(1, 0)$.

7. If $R$ is the shaded region consisting of the two disks shown below, whose boundary $\partial R$ is oriented as shown, evaluate $\int_{\partial R} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (2x^2 - 3y)\mathbf{i} + (5x + y^{10})\mathbf{j}$.

8. Let $R$ be the top half of the ellipse $x^2 + \frac{y^2}{4} = 1$.
   a) Find a counterclockwise parameterization of the boundary $\partial R$ of $R$.
   b) Compute the double integral $\iint_R 3x^2 y \, da$. Hint: Can you find a vector function $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2 y$?

9. Let $C$ be the line segment from $(a, b)$ to $(c, d)$, where $a < c$ and $b > d$. Compute $I_1 = \int_C e^{-x} \, dx$, $I_2 = \int_C e^{-y} \, dy$ and $I_3 = \int_C e^{-s} \, ds$. Explain (by thinking about what the integrals actually represent) the signs of $I_1$, $I_2$ and $I_3$. Also explain why $I_1$ doesn’t depend on $b$ and $d$. 