This writeup should give you an idea of what I would look for when grading this problem. It is not intended to be a perfect solution. In particular, I would expect a better picture, with the planes $\theta = 0$ and $\theta = \pi/6$ labeled. While I could draw better pictures in two minutes by hand, it is quite time-consuming with the computer. I’ve chosen to leave it as is. If you are in doubt about what you should include in a good picture, talk to your workshop instructor.

#35. Find the volume of the smaller wedge cut from a sphere of radius $a$ by two planes that intersect along a diameter at an angle of $\pi/6$.

This problem is considerably easier if we put some care into our choices of spheres and planes. First, it’s generally simplest to center a sphere at the origin. Next, we should try to choose the easiest possible equations for the two planes.

Because we are working with a piece of a sphere, it seems likely that we should work with spherical coordinates; with that in mind, the simplest choices for the planes are probably those described by $\theta = 0$ and $\theta = \pi/6$. Figure 1(a) shows a sphere of radius $a$ together with these planes; we’re interested in the apple-slice-shaped piece inbetween the planes. We’ll call this solid region $W$. A closup view is shown in Figure 1(b).

![Figure 1](image1)

Figure 1. (a) The intersection of a sphere and two planes (b) The wedge $W$ cut out of the sphere
By examining the pictures, we can see that the wedge $W$ is described by the following inequalities in spherical coordinates:

\[
0 \leq \rho \leq a \\
0 \leq \theta \leq \pi/6 \\
0 \leq \phi \leq \pi
\]

Hence the volume is given by the integral

\[
\int \int \int \limits_{W} dV = \int_{0}^{\pi/6} \int_{0}^{\pi} \int_{0}^{a} \rho^2 \sin \phi \ d\rho d\phi d\theta \\
= \int_{0}^{\pi/6} \int_{0}^{\pi} \frac{a^3}{3} \sin \phi \ d\phi d\theta \\
= \int_{0}^{\pi/6} \left[ -\frac{a^3}{3} \cos \phi \right]_{0}^{\pi} \ d\theta \\
= \int_{0}^{\pi/6} \frac{2a^3}{3} \ d\theta \\
= \frac{2\pi a^3}{18} = \frac{\pi a^3}{9}
\]

If we had let $\theta$ range from 0 all the way to $2\pi$, we would have obtained the volume of the full sphere, which is $\frac{4}{3}\pi a^3$. This observation leads to a quick way to check our answer geometrically. Since $\pi/6$ is one twelfth of $2\pi$, we should expect the wedge to have one twelfth of the volume of the sphere. Indeed,

\[
\frac{1}{12} \cdot \frac{4}{3} \pi a^3 = \frac{\pi a^3}{9}
\]

as desired.