Overview: Limits and Continuity

This review sheet is not meant to be your only form of studying. Understanding all the homework problems and lecture material are essential for success in the course. This review sheet only contains the key ideas of these sections.

2.2-2.3: Limits. Find \( \lim_{x \to a} f(x) \), where \( a \) is finite number.

1. Understand the various algebraic methods to deal with different limits problems. There are some common strategies:

   - Factor: Ex: 11-16 in Sec. 2.3.
   - Expand: Ex: 17-20 in Sec. 2.3.
   - Multiply conjugates: EX: 21, 22, 25 in Sec. 2.3.
   - \( f \) is continuous at \( a \), then \( \lim_{x \to a} f(x) = f(a) \).

   Useful formula:
   \[
   a^2 - b^2 = (a + b)(a - b)
   \]

2. Using The Squeeze Theorem to find limit:

   **Theorem 0.1. The Squeeze (Sandwich) Theorem**
   If \( f(x) \leq g(x) \leq h(x) \) for \( x \) near \( a \) (except possibly at \( a \)) and \( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \), then
   \[
   \lim_{x \to a} g(x) = L.
   \]

   EX: 36-40 in Sec. 2.3 and 36, 37 in Sec. 2.6.

3. Simplify the absolute values: EX: 41-46 in Sec. 2.3.

2.5: Continuity. \( \lim_{x \to a} f(x) = f(a) \).

1. Definition of Continuity:

   \( f(x) \) is continuous at \( a \) if \( \lim_{x \to a} f(x) = f(a) \). This requires
   
   (a) \( f(x) \) is defined at \( a \).
   
   (b) \( \lim_{x \to a} f(x) \) exists
   
   (c) \( \lim_{x \to a} f(x) = f(a) \).

2. \( f(x) \) is continuous from the right at \( a \) if \( \lim_{x \to a^+} f(x) = f(a) \).

3. \( f(x) \) is continuous from the left at \( a \) if \( \lim_{x \to a^-} f(x) = f(a) \).

Theorem 0.2. (The Intermediate Value Theorem)
\( f \) is continuous on the closed interval \([a, b]\). Let \( N \) be any number between \( f(a) \) and \( f(b) \), where \( f(a) \neq f(b) \). Then we can find a number \( c \) in \((a, b)\) such that \( f(c) = N \).

EX: See the Sample problems for Quiz 2.

2.6 : Limits at infinity. Find \( \lim_{x \to \infty} f(x) \) or \( \lim_{x \to -\infty} f(x) \).

1. Find horizontal asymptotes: Compute \( \lim_{x \to \infty} f(x) = L \) and \( \lim_{x \to -\infty} f(x) = R \). Then \( y = L \) and \( y = R \) are horizontal asymptotes of \( f \). EX: 41-46 in Sec. 2.6.

2. Find vertical asymptotes: Observe what \( x \) will make \( f(x) \) goes to \( \infty \) or \( -\infty \). EX: if \( f(x) = \frac{4x^2 + 1}{4x - 7} \), then \( x = \frac{4}{7} \) is the vertical asymptote. EX: 41-46 in Sec. 2.6.

3. Some special functions:

\[
\begin{align*}
\lim_{x \to \infty} e^{-x} &= 0, & \lim_{x \to \infty} e^x &= \infty, & \lim_{x \to \infty} \arctan(x) &= \frac{\pi}{2}, & \lim_{x \to -\infty} \arctan(x) &= -\frac{\pi}{2} \\
\lim_{x \to 0} \arctan(x) &= 0, & \lim_{x \to 1} \arctan(x) &= \frac{\pi}{4}, & \lim_{x \to \sqrt{3}} \arctan(x) &= \frac{\pi}{3}, \ldots
\end{align*}
\]

EX: 33, 34, 37, 38 in Sec. 2.6.

4. Limits of a composition:

EX:1 \( \lim_{x \to \infty} x \sin \left(\frac{1}{x}\right) \).

Ans: Let \( u = \frac{1}{x} \). Then \( u = \frac{1}{x} \to 0 \) as \( x \to \infty \). Therefore,

\[
\lim_{x \to \infty} x \sin \left(\frac{1}{x}\right) = \lim_{x \to \infty} \sin \left(\frac{1}{x}\right) = \lim_{u \to 0} \sin \left(\frac{1}{x}\right) = 1.
\]

EX:2 \( \lim_{x \to \infty} \sin \left(\frac{\pi(x^2 + x)}{4x^2 + 3}\right) \).

Ans: Let \( u \) be the inner function, that is,

\[
u = \frac{\pi(x^2 + x)}{4x^2 + 3} = \pi \frac{x^2 + x}{4x^2 + 3} = \pi \frac{1 + \frac{1}{x}}{4 + \frac{3}{x^2}} \to \frac{\pi}{4} \quad \text{as} \quad x \to \infty.
\]

Therefore, \( u \to \frac{\pi}{4} \) as \( x \to \infty \). And we obtain

\[
\lim_{x \to \infty} \sin \left(\frac{\pi(x^2 + x)}{4x^2 + 3}\right) = \lim_{u \to \frac{\pi}{4}} \sin(u) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.
\]

EX:3 \( \lim_{x \to \infty} \arctan \left(\frac{1}{x^2}\right) \).

Ans: Let \( u \) be the inner function, that is,

\[
u = \frac{1}{x^2} \to 0 \quad \text{as} \quad x \to \infty.
\]

Therefore,

\[
\lim_{x \to \infty} \arctan \left(\frac{1}{x^2}\right) = \lim_{u \to 0} \arctan(u) = \arctan(0) = 0.
\]