Review:

A function $f$ is continuous at $a$ if \( \lim_{x \to a} f(x) = f(a) \).

**EX:** \( f(x) = \begin{cases} e^x & \text{if } x < 0 \\ 9x^2 + x + 1 & \text{if } 0 \leq x \leq 2 \end{cases} \)

Is $f$ continuous at $x = 0$?

**Ans:**

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^x = 1 \quad (e^0 = 1)
\]

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (9x^2 + x + 1) = 1
\]

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 1 = f(0)
\]

So, $f$ is continuous at $0$. 
EX: \( f(x) = \frac{x^3 + 7}{x^2 - 2x - 3} \). Where is \( f \) continuous?

Ans: \( x^2 - 2x - 3 = 0 \)  \( \Rightarrow \) \((x - 3)(x + 1) = 0\)

\( \Rightarrow \) \( x = 3, -1 \)

So, \( f \) is continuous at every \( x = a \) except \( x = 3, -1 \).

Thm: (R123 in the Book)
Functions are continuous at every number \( a \) in their domains.

- Polynomials
- Rational functions
- Trig. functions
- \( \ldots \) etc. Ex: \( e^x \).

EX: Find \( \lim_{x \to 0} \frac{1 - \cos x}{x \sin x} \).

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= \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{x \sin x (1 + \cos x)}

= \lim_{x \to 0} \frac{1 - \cos^2 x}{x \sin x (1 + \cos x)}

= \lim_{x \to 0} \frac{\sin^2 x}{x \sin x (1 + \cos x)}

= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x}

= \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{1 + \cos x}

= 1 \cdot \frac{1}{2} = \frac{1}{2}
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$\sin^2 \theta + \cos^2 \theta = 1$
\[ \text{Thm.1: } f \text{ is continuous at } b, \lim_{x \to a} g(x) = b. \]

Then \[ \lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right). \]

After adding continuity condition to function \( g \), we have.

\[ \text{Thm.2: } g \text{ is continuous at } a, f \text{ is continuous at } g(a). \text{ Then } (f \circ g)(x) = f(g(x)) \text{ is continuous at } a. \]

Moreover, \[ \lim_{x \to a} f(g(x)) = f(g(a)). \]

\[ \text{EX: Find } \lim_{x \to 3} \frac{\sin(x-3)}{x-3}. \]

\[ \text{Ans: let } \begin{cases} g(x) = x - 3, & \text{then } g \text{ is continuous at } 0. \\ f(t) = \begin{cases} \frac{\sin(t)}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases} \end{cases} \]

\[ \lim_{t \to 0} f(t) = \lim_{t \to 0} \frac{\sin(t)}{t} = 1 = f(0). \]

So, \( f \) is continuous at 0.

By Thm.2, \( (f \circ g)(x) \) is continuous at \( x = 3 \).

So, \[ \lim_{x \to 3} (f \circ g)(x) = (f \circ g)(3) \]
\[ = f(0) \]
\[ = 1 \]
\[ = g(3) \]
Another approach:

let \( u = x - 3 \). Observe that as \( x \to 3 \Rightarrow u \to 0 \).

Substitute \( u \) for \( x - 3 \),

\[
\lim_{x \to 3} \frac{\sin(x-3)}{x-3} = \lim_{u \to 0} \frac{\sin(u)}{u} = 1.
\]

Ihm, (Intermediate Value Theorem)

\( f \) is continuous on closed interval \([a, b]\). If \( N \) is any number between \( f(a) \) and \( f(b) \), \( f(a) \neq f(b) \). Then there exists a number \( c \in (a, b) \) such that \( f(c) = N \).

Fig. 1

Fig. 2
Ex: Show there is a root of \(4x^3 - 6x^2 + 3x - 2 = 0\) between 1 and 2.

Ans: \(f(x) = 4x^3 - 6x^2 + 3x - 2\) is continuous on \((-\infty, \infty)\)

\(f(1) = 4 - 6 + 3 - 2 = -1 < 0\)

\(f(2) = 32 - 24 + 6 - 2 = 12 > 0\)

\(f(1) < 0 < f(2)\). By Intermediate Value Thm, we can find \(1 < c < 2\) such that \(f(c) = 0\).

Ex: Show \(x^2 + \sin x = 2\) has a solution between 0 and \(\frac{\pi}{2}\).

Ans: Let \(f(x) = x^2 + \sin x - 2\). Then \(f\) is continuous on \((-\infty, \infty)\)

\(f(0) = 0 + \sin 0 - 2 = -2\)

\(f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} + \sin \left(\frac{\pi}{2}\right) - 2 = \frac{\pi^2}{4} + 1 - 2 = \frac{\pi^2}{4} - 1 > 0\)

\(f(0) < 0 < f\left(\frac{\pi}{2}\right)\)

By Intermediate Value Thm, we can find \(0 < c < \frac{\pi}{2}\) such that \(f(c) = 0\).
Hence \(f(c) = c^2 + \sin c - 2 = 0\).
§ 2.6 Limits at Infinity, Horizontal Asymptotes

In Section 2.2, we investigated infinite limits and vertical asymptotes.

**Ex:** \( \lim_{x \to 0} \frac{1}{x^2} = \infty \) \quad x \to 0, \text{ vertical asymptote}

In Sec 2.6, we let \( x \) become arbitrarily large (positive or negative) and see what happens to \( f(x) \).

\[ f(x) = \frac{x^2 - 1}{x^2 + 1} \]

\[ \lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1 \quad ; \quad \lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1 \]

\( f(x) \) can be arbitrarily close to 1 by taking \( x \) sufficiently large.

Another notation:

\[ \frac{x^2 - 1}{x^2 + 1} \to 1 \quad \text{as} \quad x \to \infty \quad \frac{x^2 - 1}{x^2 + 1} \to 1 \quad \text{as} \quad x \to -\infty \]

\( y = 1 \) is horizontal asymptote
**Definition:** The line \( y = L \) is called a "horizontal asymptote" of the curve \( y = f(x) \) if either \( \lim_{x \to \infty} f(x) = L \) or \( \lim_{x \to -\infty} f(x) = L \).

**EX:** Find the horizontal asymptote of \( y = \tan^{-1}x \).

(Note: \( \tan^{-1}x = \arctan x \))

**Ans:**

\[
\lim_{x \to \infty} \tan^{-1}x = \frac{\pi}{2}
\]

\[
\lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2}
\]

So \( y = \frac{\pi}{2}, y = -\frac{\pi}{2} \) are horizontal asymptotes.

In fact, \( x = \pm \frac{\pi}{2} \) are vertical asymptotes of \( \tan \).