§ 3.5 Implicit Differentiation.

The functions we met before can be expressed as \( y = f(x) \). e.g. \( y = \frac{1}{x}, \quad y = \sqrt{x^2 + 1} \).

However, some functions are defined implicitly by a relation between \( x, y \) such as

\[
x^2 + y^2 = 25, \quad y = \pm \sqrt{25 - x^2}.
\]

\[
x^3 + y^3 = 6xy.
\]

It is hard to solve an equation for \( y \) in terms of \( x \) in order to find \( \frac{dy}{dx} \). So we need a new method called implicit differentiation to find \( \frac{dy}{dx} \).

**EX1:** \( x^2 + y^2 = 25 \). Find \( \frac{dy}{dx} \).

*Ans.* \[
\frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} (25)
\]

\[
2x + \frac{d}{dx} (y^2) = 0
\]

\[
\frac{d}{dx} (y^2) = \frac{d}{dy} (y^2) \frac{dy}{dx} \quad \text{(by chain rule)}
\]

\[
= 2y \frac{dy}{dx}
\]

So,

\[
2x + 2y \frac{dy}{dx} = 0
\]

\[
\Rightarrow \frac{dy}{dx} = -\frac{x}{y}.
\]
§ Implicit Differentiation.

We can use the chain rule to find the derivative of an implicit defined function without having to obtain an explicit expression for the function.

**Ex. 2:** \( xy = 1 \). Find \( \frac{dy}{dx} \) by using implicit differentiation.

\[ \text{Ans: } \frac{d}{dx}(xy) = \frac{d}{dx}(1) = 0 \]

\[ \frac{dx}{dx} y + x \frac{dy}{dx} = 0 \]

\[ y + x \frac{dy}{dx} = 0 \]

\[ \frac{dy}{dx} = -\frac{y}{x}. \]

(b) Finding \( \frac{dy}{dx} \) by using power rule (the basic way).

\[ y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left( x^{-1} \right) \]

\[ = -x^{-2} \]

\[ = -\frac{1}{x^2}. \]

(c) Compare (a)’s and (b)’s results.

\( (a) = -\frac{y}{x} \quad \frac{y}{x} = \frac{1}{x} \Rightarrow -\frac{1}{x} = -\frac{1}{x^2} = (b) \)

They are the same.
EX 3. \( x^3 + y^3 = 6xy \). Find \( \frac{dy}{dx} \).

**Ans:** \( \frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(6xy) \)

\[ 3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx} \]

\[ (3y^2 - 6x) \frac{dy}{dx} = 6y - 3x^2 \]

\[ \frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x} \]

(b) Find the tangent line to this curve at \((3, 3)\).

**Ans:** Tangent line equation:

\[ y - 3 = y'(3) (x - 3) \]

\[ y'(3) = \left. \frac{2y - x^2}{y^2 - 2x} \right|_{x=3, y=3} = \frac{6 - 9}{9 - 6} = -1 \]

\[ y - 3 = -1(x - 3) \]

\[ y = x \]
EX 4: $\sqrt{x} + \sqrt{y} = 3$. Find $y'$.

\[
x^{\frac{1}{2}} + y^{\frac{1}{2}} = 3.
\]

\[
\frac{d}{dx} \left( x^{\frac{1}{2}} \right) + \frac{d}{dx} \left( y^{\frac{1}{2}} \right) = 0
\]

\[
\frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} y^{-\frac{1}{2}} y' = 0
\]

\[
x^{-\frac{1}{2}} + y^{-\frac{1}{2}} y' = 0.
\]

\[
y' = -y^{\frac{1}{2}} x^{-\frac{1}{2}}.
\]

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EX 5: $x^2 + 2xy - y^2 + x = 2$. Find tangent line equation at (1, 2).

Ans: 

\[
2x + 2y + 2xy' - 2yy' + 1 = 0
\]

\[
y' = \frac{-2x - 2y - 1}{2x - 2y}
\]

\[
\left. \frac{dy}{dx} \right|_{(1,2)} = \frac{-2 - 4 - 1}{2 - 4} = \frac{-7}{-2} = \frac{7}{2}
\]

\[
y - 2 = \frac{7}{2} (x - 1)
\]
EX6: Find $y''$ if $x^2 + y^2 = 25$.

From EX 1, we knew

$$y' = \frac{dy}{dx} = -\frac{x}{y}. \quad (1)$$

To find $y''$, remember $y$ is a function of $x$.

$$y'' = \frac{d}{dx}(-\frac{x}{y}) \quad (2 \text{ Quotient rule})$$

$$= -\frac{\frac{d}{dx}(x) y - x \frac{d}{dx}(y)}{y^2}$$

$$= -\frac{y - x y'}{y^2}$$

Using (1)

$$y'' = -\frac{y - x (-\frac{x}{y})}{y^2}$$

$$= -\frac{y + \frac{x^2}{y}}{y^2}$$

$$= -\frac{y^2 + x^2}{y^3}$$

$$= -\frac{25}{y^3} \quad \#$$
§ Derivative of Inverse Trig. Functions.

1. \[ \frac{d}{dx} (\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \]

\[ y = \sin^4 x \] means \( \sin y = x \), and \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \).

\[ \sin y = x. \] 2 implicit differentiation

\[ \cos y \frac{dy}{dx} = 1 \]

\[ \frac{dy}{dx} = \frac{1}{\cos y} \]

Now \[ \cos y = -\sqrt{1 - \sin^2 y} \] since \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \).

\[ = \sqrt{1 - x^2} \]

So, \[ \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}} \]

2. \[ \frac{d}{dx} (\tan^{-1}x) = \frac{1}{1+x^2} \]

3. \[ \frac{d}{dx} (\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \]

4. \[ \frac{d}{dx} (\sec^{-1}x) = -\frac{1}{x \sqrt{x^2-1}} \]

5. \[ \frac{d}{dx} (\sec^{-1}x) = \frac{1}{x \sqrt{x^2-1}} \]

6. \[ \frac{d}{dx} (\cot^{-1}x) = -\frac{1}{1+x^2} \]

The Box in Page 214 in textbook.