The tangent line at \((a, f(a))\) is an approximation to the curve \(y = f(x)\) when \(|x-a|\) is small.

We know the tangent line equation at \((a, f(a))\) is

\[ y - f(a) = f'(a)(x-a) \]

or

\[ y = f(a) + f'(a)(x-a). \]

\[ L(x) = f(a) + f'(a)(x-a) \] is the approximation of \(f\) when \(|x-a|\) is small, i.e.,

\[ f(x) \approx L(x) = f(a) + f'(a)(x-a), \]

\(L(x) = f(a) + f'(a)(x-a)\) is called the linearization of \(f\) at \(a\).
EX: Find the linearization of $f(x) = \sqrt{x+3}$ at $a = 1$. Use it to approximate the number $\sqrt{3.98}$.

Ans: Linearization $L(x) = f(1) + f'(1)(x-1)$

$f(1) = \sqrt{4} = 2$.

$f'(x) = \frac{1}{2} (x+3)^{-\frac{1}{2}} \Rightarrow f'(1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

$L(x) = 2 + \frac{1}{4} (x-1) = \frac{1}{4} x + \frac{7}{4} \Rightarrow \sqrt{x+3} \approx \frac{1}{4} x + \frac{7}{4}$

$\sqrt{3.98} \approx \frac{1}{4} (0.98) + \frac{7}{4} = 1.995$.

An alternative notation to linear approximation is using differentials $dx$ and $dy$.

$dy = f'(x) \, dx$.

Here $dx$ is an independent variable.

$dy$ is a dependent variable, since it depends on $x$, $dx$.

The differential $dy$ at $a$ is $dy = f'(a) \, dx$. 3.10-2
Linear approximation

\[ f(x) \approx f(a) + f'(a) (x-a) \]

can be written as

\[ f(a+dx) \approx f(a) + f'(a) \, dx \]

\[ dy \]

\[ \Rightarrow f(a+dx) \approx f(a) + dy. \]

\[ \text{§} \quad dx, \, dy \, \text{vs.} \, \Delta x, \, \Delta y \]

When \( x \) is changed by an amount \( \Delta x \),
then \( \Delta y = f(x+\Delta x) - f(x). \) (this value is
difficult to compute in practice).

But, we can compute an approximate value
of this change \( \Delta y \). The approximate value
is

\[ dy = f'(x) \, dx. \]
EX 1: \( f(x) = \sqrt[3]{x} \). Compute \( \Delta y = f(1.1) - f(1) \) and the corresponding \( dy \), i.e., \( dy \) at \( a = 1 \), where \( dx = \Delta x = 0.1 \).

\[ \begin{align*}
\text{Ans: } & 1) \ \text{Using calculator, we find } f(1.1) = \sqrt[3]{1.1} \approx 1.032. \\
& f(1) = 1, \text{ so,} \\
& \Delta y = f(1.1) - f(1) \\
& = 0.032. \\
2) \ dy = f'(x) \, dx = \frac{1}{3} x^{-\frac{2}{3}} \, dx \\
& \text{When } a = 1, \ dx = \Delta x = 0.1, \text{ then } \\
& dy = \frac{1}{3} \cdot (0.1) \\
& = 0.033. \#
\end{align*} \]

EX 2: Use a differential to find an approximate value of \( \sqrt{69} \).

\[ \text{Ans: We know the exact value of } \sqrt{64} = 8. \text{ So we need to find an approximate value of } \Delta y = \sqrt{69} - \sqrt{64}. \]

The function \( f(x) = \sqrt{x} \).
\[
\frac{dy}{dx} = f'(x) dx \\
= \frac{1}{2\sqrt{x}} dx.
\]

When \( x = 64 \), \( dx = \Delta x = 67 - 64 = 3 \)
\[
\frac{dy}{dx} = \frac{1}{2\sqrt{64}} (3) = \frac{3}{16}
\]

Hence
\[
\sqrt{67} - \sqrt{64} \approx dy = \frac{3}{16}
\]
\[
\sqrt{67} \approx \sqrt{64} + \frac{3}{16} = 8 + \frac{3}{16} = \frac{131}{16}
\]
Ex: Find the differential \( dy \) and evaluate \( dy \) for the given \( x \) and \( dx \).

\[ y = \cos(\pi x), \quad x = \frac{1}{3}, \quad dx = -0.02. \]

Ans: \( dy = f'(x) \, dx \).

\[ = -\sin(\pi x) \pi \, dx \]

\[ dy = -\sin\left(\frac{\pi}{3}\right) \pi (-0.02) \]

\[ = \frac{\sqrt{3}}{2} \pi \cdot (0.02) \]

\[ = (0.01)\sqrt{3} \pi. \]