§ 4.1

Critical number: \( f'(c) = 0 \) or \( f'(c) \) does not exist.

Closed Interval method: (Find absolute max / min.)

1. \( f \) is continuous on \([a, b]\).
2. Find critical numbers in \((a, b)\).
3. Evaluate \( f \) at critical numbers.
4. Evaluate \( f \) at endpoints.
5. Compare those values in 3, 4.

§ 4.2

Mean Value Theorem:

1. \( f \) is continuous on \([a, b]\).
2. \( f \) is differentiable on \((a, b)\).

Then there is a number \( c \) in \((a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

§ 4.3

Increasing / decreasing Test.

\( f'(x) > 0 \) on an interval, then \( f \) is increasing on that interval.

\( f'(x) < 0 \), then \( f \) is decreasing.
Consider EX 1. We discussed last time.

\[ f(x) = 4x^3 + 3x^2 - 6x + 1 \]

has critical numbers \( x = \frac{1}{2}, -1 \).

Q: How do we know if \( f \) has local max or min at \( \frac{1}{2}, -1 \)?

\[
\begin{array}{ccc}
\text{\( f \) is increasing} & \text{\( f \) is decreasing} & \text{\( f \) is increasing} \\
f' > 0 & f' < 0 & f' > 0 \\
-1 & \frac{1}{2} & 1
\end{array}
\]

Ans: \( f(-1) \) is local max value since \( f \) changes from positive to negative.

\( f \left( \frac{1}{2} \right) \) is min value since \( f \) changes from negative to positive.
Recall that if \( f \) has local max or min at \( c \), then \( c \) is a critical number of \( f \). (Fermat's Theorem)

But not every critical number gives rise to a max or min.

The following test tells us when \( f \) has a local max or min at a critical number \( c \).

The first derivative test:

If \( c \) is a critical number of a continuous function \( f \):

1. \( f'(x) \) changes from positive to negative at \( c \), then \( f(c) \) is a local maximum value.

2. \( f'(x) \) changes from negative to positive at \( c \), then \( f(c) \) is a local minimum value.

3. \( f' \) does not change sign at \( c \), then \( f \) has no local maximum or minimum at \( c \).

Ex: See Fig. 2.
What does $f''$ can tell us about $f$?

(a) Concave upward

(b) Concave downward.

Concavity Test (an interval)

1. $f'' > 0$ for all $x \in I$, then the graph of $f$ is concave upward on $I$.

2. $f'' < 0$ for all $x \in I$, then the graph of $f$ is concave downward on $I$.

Observation:

In Fig. 3 (a), $f'$ is increasing, thus $f'' > 0$.

In Fig. 3 (b), $f'$ is decreasing, thus $f'' < 0$.
An inflection Point.

A point on the graph of \( f \) is called an inflection point if \( f \) is continuous there and turns from concave upward to concave downward, or vice versa.

\( y = x^3 \)

\( (0,0) \) is an inflection point.
**EX 2:** Let \( f(x) = -x^4 - \frac{8}{3} x^3 + 6x^2 \).

Find the intervals on which \( f(x) \) is concave upward and concave downward.

**Ans:**
\[
\begin{align*}
f'(x) &= -4x^3 - 8x^2 + 12x \\
f''(x) &= -12x^2 - 16x + 12 \\
      &= -4(3x^2 + 4x - 3)
\end{align*}
\]

To find intervals on which \( f'' > 0 \) and \( f'' < 0 \).

\[
3x^2 + 4x - 3 = 0
\]
\[
x = \frac{-4 \pm \sqrt{16 + 36}}{6} = \frac{-4 \pm \sqrt{52}}{6} = \frac{-4 \pm 2\sqrt{13}}{6} = \frac{-2 \pm \sqrt{13}}{3}
\]

\[
\Rightarrow C_1 = \frac{-2 - \sqrt{13}}{3}, \quad C_2 = \frac{-2 + \sqrt{13}}{3}
\]

<table>
<thead>
<tr>
<th>( f'' )</th>
<th>( f'' &gt; 0 )</th>
<th>( f'' &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\infty, C_1)</td>
<td>( C_1, C_2 )</td>
<td>( C_2, \infty )</td>
</tr>
</tbody>
</table>

\[
f''(1) = -12 - 16 + 12 = -16 < 0 \Rightarrow (C_2, \infty) \text{ concave down}
\]
\[
f''(0) = 12 > 0 \Rightarrow (C_1, C_2) \text{ concave up}
\]
\[
f''(-2) = -4 < 0 \Rightarrow (-\infty, C_1) \text{ concave down}
\]

\[4.3-6]
The second derivative test:

- $f''$ is continuous near $c$.

1. If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has local min at $c$.

2. If $f'(c) = 0$, and $f''(c) < 0$, then $f$ has local max at $c$. 
**EX3:** \( f(x) = -x^4 - \frac{8}{3}x^3 + 6x^2. \) Use the second derivative to find local max/min.

**Ans:**

\[
f' = -4x^3 - 8x^2 + 12x
\]

\[
= -4x(x^2 + 2x - 3)
\]

\[
= -4x(x - 1)(x + 3)
\]

\( x = 0, 1, -3 \) critical numbers.

Since \( f'' \) is polynomial, \( f'' \) is continuous on \((\infty, \infty)\).

Use the second derivative test:

\[
f''(-3) = -4(3.9 + (-12) - 3) = -48 < 0.
\]

\( \Rightarrow \) \( f \) has local max at \( x = -3 \).

\[
f''(0) = 12 > 0.
\]

\( \Rightarrow \) \( f \) has local min at \( x = 0 \).

\[
f''(1) = -16 < 0.
\]

\( \Rightarrow \) \( f \) has local max at \( x = 1 \).

\[
\text{You also can use the first derivative test to conclude the same result.}
\]