6x: Find the volume of the solid obtained by rotating the region bounded by \( y = x^3, \ y = 8, \ x = 0 \) about the \( y \)-axis.

**Ans:**

When we slice it at height \( y \), we get a disk with radius \( x = y^{\frac{1}{3}} \).

\[
A(y) = \left( y^{\frac{1}{3}} \right)^2 \pi = y^{\frac{2}{3}} \pi.
\]

\[
V = \int_0^8 \left( y^{\frac{2}{3}} \pi \right) \ dy
\]

\[
= \pi \left. \frac{3}{5} y^{\frac{5}{3}} \right|_0^8
\]

\[
= \pi \left( \frac{3}{5} \cdot 8^{\frac{5}{3}} \right)
\]

\[
= \pi \left( \frac{3}{5} \cdot 2^5 \right)
\]

\[
= \frac{96\pi}{5}.
\]

( Disk-shaped cross section )
Find the volume of the solid obtained by rotating the region bounded by \( y = x, \ y = x^2 \) about \( x = -1 \).

Each washer-shaped cross section

\[
\text{Any } dy = \left[ \pi (\text{outer radius})^2 - \pi (\text{inner radius})^2 \right] dy
\]

\[
= \pi \left( (1 + \sqrt{y})^2 - (1 + y)^2 \right) dy
\]

\[
V = \pi \int_0^1 \left( (1 + \sqrt{y})^2 - (1 + y)^2 \right) dy
\]

\[
= \pi \int_0^1 1 + 2\sqrt{y} + y - (1 + 2y + y^2) \ dy
\]

\[
= \pi \left[ \frac{4}{3} y^{\frac{3}{2}} - \frac{2}{3} y^2 - \frac{1}{3} y^3 \right]_0^1 = \frac{\pi}{3}.
\]
§ 6.3 Volumes by Cylindrical Shells

Some volume problems are hard to handle by the methods of slicing solid into disk/washer cross section.

3X1: Find volume of solid obtained by rotating A about y-axis.

If we slice perpendicular to y-axis, we get a washer-shaped cross section.

To find volume, we need to know outer radius and inner radius, i.e., need to solve $y = 2x^2 - x^3$ for $x$ in terms of $y$. Is it hard?

Hence we need a new method, called "method of cylindrical shells."
The method of cylindrical shells.

The solid $S$ is obtained by rotating the region bounded by $y = f(x)$, $y = 0$, $x = a$, $x = b$ about $y = ax + b$.

Volume of a cylindrical shell: $V = \text{(area of the walls)} \times \text{(thickness)}$

$$V = 2\pi x f(x) \frac{h}{\Delta x}.$$  

So, the volume of $S$ is

$$\int_a^b 2\pi x f(x) \, dx.$$
Ans. to EX1:

\[
V = \int_a^b 2\pi r h \, dx.
\]

\[
= \int_0^2 2\pi x f(x) \, dx
\]

\[
= \int_0^2 2\pi x (2x^2 - x^3) \, dx
\]

\[
= 2\pi \left( \frac{1}{2} x^4 - \frac{1}{5} x^5 \right) \bigg|_0^2
\]

\[
= \frac{16}{5} \pi.
\]