Method of Cylindrical Shells:

The volume of solid $S = V = \int_{a}^{b} (2\pi x) f(x) \, dx$

Volume of Cylindrical Shell:

$$V = \pi r^2 h \Delta x$$

$$= 2\pi r f(x) \Delta x$$
The solid $S$ is obtained by rotating the same region as $E_1$, about $x = -1$.

**Ans:**

\[
\sqrt{ } = \int_{a}^{b} 2\pi r \cdot h \, dx
\]

\[
= \int_{0}^{2} 2\pi (x+1)(2x^2-x^3) \, dx
\]

\[
= 2\pi \int_{0}^{2} 2x^3-x^4 + 2x^2 - x^3 \, dx
\]

\[
= 2\pi \left[ \frac{4}{5}x^5 - \frac{1}{5}x^5 + \frac{2}{3}x^3 \right]_0^2
\]

\[
= 2\pi \left( 4 - \frac{1}{5} \cdot 32 + \frac{16}{3} \right)
\]

\[
= 2\pi \left( \frac{44}{15} \right) = \frac{88\pi}{15}.
\]
\[ 8x^3 \] Find volume of the solid obtained by rotating the region bounded by \( y = \sqrt{x}, \ x = 0, \ y = 2 \) about \( x \)-axis.

**Ans:**

1. By cylindrical shells:

\[
\sqrt{\pi} = \int_0^2 2\pi x h \, dy
\]
\[
= \int_0^2 2\pi y(y^2) \, dy
\]
\[
= 8\pi.
\]
2. By slicing it, we get a washer-
shaped cross section.

\[ V = \pi \int_0^4 (outer radius)^2 - (inner radius)^2 \, dx \]

\[ = \pi \int_0^4 \left( 2^2 - (\sqrt{x})^2 \right) \, dx \]

\[ = \pi \left( 4x - \frac{1}{3}x^2 \right) \bigg|_0^4 \]

\[ = \frac{8\pi}{3} \]
**Note:** If you know how to set up an integral for the volume of a solid by knowing the concepts behind the method of slicing and the method of cylindrical shells, it is not necessary to memorize the following:

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<th>Rotate about $x$-axis</th>
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<td>Slicing</td>
<td>$V = \int_a^b A(x) , dx$</td>
<td>$V = \int_a^b 2\pi y f(y) , dy$</td>
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<tr>
<td>Shells</td>
<td>$V = \int_a^b 2\pi y f(y) , dy$</td>
<td>$V = \int_a^b 2\pi x f(x) , dx$</td>
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6.5 Average Value of a Function

Recall: To calculate the average value of finitely many values \( y_1, \ldots, y_n \).

\[
\text{Average} = \frac{y_1 + \cdots + y_n}{n}
\]

Q: How do we evaluate the average value of infinitely many data \( y_1, y_2, \ldots \)?

Let's consider that we want to compute the average value of a function \( y = f(x) \), \( a \leq x \leq b \).

- Divide \([a, b]\) into \( n \) equal subintervals, \( \Delta x = \frac{b-a}{n} \).
- Choose \( x_i^* \) in \([x_{i-1}, x_i]\), \( 1 \leq i \leq n \).
- The average of \( f(x_1^*), \ldots, f(x_n^*) \) is

\[
\frac{f(x_1^*) + \cdots + f(x_n^*)}{n} \quad \Rightarrow \quad \Delta x = \frac{b-a}{n}
\]

\[
\int_a^b f(x) \, dx = \frac{b-a}{\Delta x} \left( f(x_1^*) \Delta x + \cdots + f(x_n^*) \Delta x \right)
\]

- Take limit \( n \to \infty \),

\[6.5-1\]
The average value of $f$ on $[a,b]$ is

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) \, dx.$$  

**EX:** Find the average value of $f(x) = 1 + x^2$ on $[-1,2]$.

**Ans:**

$$f_{\text{ave}} = \frac{1}{2 - (-1)} \int_{-1}^2 1 + x^2 \, dx$$

$$= \frac{1}{3} \left[ x + \frac{x^3}{3} \right]_{-1}^2$$

$$= \frac{1}{3} \left[ (2 + \frac{8}{3}) - (-1 - \frac{1}{3}) \right]$$

$$= 2.$$
§ The Mean Value Theorem for Integrals

If \( f \) is continuous on \([a, b]\), then there exists a number \( c \) in \([a, b]\) such that

\[
f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

**Example:** \( f(x) = 1 + x^2 \) is continuous on \([-1, 2]\), the MVT for integrals says there is a number \( c \) in \([-1, 2]\) such that

\[
\int_{-1}^{2} f(x) \, dx = \frac{1}{2} \int_{-1}^{2} f(x) \, dx
\]

By previous ex.

\[
c = \frac{1}{2}
\]

Recall: MVT for derivatives. (§ 4.2)

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

6.5-3