(F09, #7) Related Rates.

Set up an equation.
Differentiate both sides with respect to $t$. (time)

\[ V = \pi r^2 h \]
\[ \frac{dV}{dt} = \pi r^2 \frac{dh}{dt} \]

Given:
\[ h = 20 \text{ cm} \]
\[ \frac{dy}{dt} = 2 \text{ cm/s} \]

Differentiate

\[ \frac{dV}{dt} = 2 \pi r \frac{dy}{dt} \]

\[ = 2 \cdot 2 \pi \cdot 2 \cdot 20 \]
\[ = 160 \pi \text{ cm}^3/\text{s} \]

Ans:

\[ \text{volume} \]
\[ \sqrt{V} = r^2 \pi h \]

\[ \text{cylinder} \]
Ans: \[ y = x (1-x^2) \]

\[ \frac{dy}{dt} = \frac{dx}{dt} (1-x^2) + x (-2x) \frac{dx}{dt} \]

\[ 4 = \frac{dx}{dt} \left( 1-1^2 \right) + (-2) \frac{dx}{dt} \]

\[ \frac{dx}{dt} = -2 \text{ (m/s)} \]
1. (F08, 12) Find an absolute maximum of the function \( f(x) = 2 \ln x + 3x - x^2 \) on \( x > 0 \).

2. (F08, 20, 21) Find the intervals where \( f(x) = (x - 3)^2(x + 3) \) is increasing and where it is decreasing. Indicate the points of a local maximum and local minimum of \( f \). Find the intervals where \( f \) is concave up and where it is concave down. Indicate the inflection points of \( f \).

3. (F09, 10) Let \( f(x) = x^3 - 3x^2 - 9x \) with domain \(-\infty < x < \infty\). Does \( f \) have any absolute max/min or local max/min?

4. (F09, 11) Let \( f(x) = 5x^4 - x^5 \) with domain \(-\infty < x < \infty\). Find the intervals where \( f \) is concave up and concave down.

5. (F09, 17) Let \( x_1 \) be an approximation of the root of \( e^{-x} - x = 0 \). Use Newton’s method to set up the expression for the \( x_2 \), in terms of \( x_1 \).

6. (F08, 11) Find \( f'(x) \) if \( f(x) = \int_{\ln x}^{1} ye^y dy \).

7. (F08, 18) Find \( F(x) \), if \( F(0) = 0 \) and \( F'(x) = \frac{x}{3 + 4x^2} \).

8. (F09, 14) Compute
\[
\int_{0}^{\ln x} \sin(e^{2t})e^{2t}dt
\]

9. (F09, 19) Compute
\[
\lim_{n \to \infty} \frac{\pi}{n} \sum_{i=1}^{n} \sin \left( \frac{\pi i}{n} \right)
\]

10. (F09, 21) Find the area enclosed by the curve \( y = |x^3 - x^2 - 2x| \) and the \( x \)-axis, between the lines \( x = -2 \) and \( x = 2 \).

11. (F07, 20) Find the area of the rectangle having the largest area that can be inscribed in a semicircle of radius \( R \).
Using critical numbers.

\[ f(x) = \frac{2}{x} + 3 - 2x, \quad x > 0 \]

\[ = \frac{2 + 3x - 2x^2}{x} = \frac{-(2x+1)(x-2)}{x}, \quad x > 0 \]

Critical numbers:

\[ +1, -\frac{1}{2}, 0, 2 \]

\[ f(2) = 2\ln 2 + 6 - 4 \]

\[ = 2\ln 2 + 2 \]
\[ f'(x) = 2(x-3)(x+3) + (x-3)^2 = (x-3)(2x+6+x-3) = (x-3)(3x+3) = 0 \]
\[ x = 3, \ -1 \]

\[ \begin{array}{c|c|c|c}
& -1 & 3 & \\
\hline
f' & + & + & \\
\end{array} \]

\[ f''(x) = 3x+3 + (x-3) \cdot 3 = 6x - 6 = 0 \]
\[ x = 1 \]

\[ \begin{array}{c|c|c|c}
& - & + & \\
\hline
f'' & - & + & \\
\end{array} \]

\( f \) is increasing on \((3,\infty), (-\infty, -1)\)
\( f \) is decreasing on \((-1, 3)\)
\( f \) is concave up on \((1, \infty)\)
\( f \) is concave down on \((-\infty, 1)\)

\( f' \) changes from positive to negative, \( f(-1) \) local max.
\( f' \) changes from negative to positive, \( f(3) \) local min.
\( f'' \) changes from negative to positive, \( x=1 \) inflection pt.
\[
\begin{align*}
\frac{d}{dx} f(x) &= 3x^2 - 6x - 9 \\
&= 3(x^2 - 2x - 3) \\
&= 3(x - 3)(x + 1) = 0 \\

+ & & - & & + \\
-1 & & 1 & & 3
\end{align*}
\]

\[
f''(x) = 6x - 6 = 0 \implies x = 1
\]

\(f\) concave up if \(x > 1\)

\(f\) concave down if \(x < 1\)

\(\implies f\) has local max at \(x = -1\)

\(\min\) at \(x = 3\)

Since \(\lim_{x \to -\infty} f(x) = +\infty\), \(f(x)\) has no

\(\lim_{x \to -\infty} f(x) = -\infty\) \(\quad\) \(\text{global max/min}\).
\( f'(x) > 0 \iff f \text{ is concave up} \)
\( f''(x) < 0 \iff f \text{ is concave down} \)

**Ans:**

\[
\begin{align*}
  f(x) &= 20x^3 - 5x^4 \\
  f''(x) &= 60x^2 - 20x^3 \\
          &= 20x^2(3 - x)
\end{align*}
\]

\[
\begin{array}{c|c|c|c}
0 & + & 3 & - \\
\hline
\end{array}
\]

\( f \) is concave up on \((0, 3)\)
\( f \) is concave down on \((-\infty, 0) \cup (3, \infty)\)
(F09, #17) Newton's Method.

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

\[ f(x) = e^{-x} - x \]

\[ f'(x) = -e^{-x} - 1 \]

\[ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \]

\[ = x_1 - \frac{e^{-x_1} - x_1}{-e^{-x_1} - 1} \]

\[ \# \]
\[(F08, \#11) \quad FTC, \; part \; 1.\]

\[f'(x) = \frac{d}{dx} \left( \int_{\ln x}^{y} e^y \, dy \right) \]
\[= - \frac{1}{x} \ln x \cdot e^{\ln x} = - \frac{1}{x} \ln x \cdot x \]
\[= - \ln x.\]

\[(F08, \#18) \quad \text{Antiderivative and u-substitution.}\]

\[F(x) = \int \frac{x}{3 + 4x^2} \, dx = \int \frac{1}{u} \cdot \frac{du}{8} \quad u = 3 + 4x^2 \]
\[= \frac{1}{8} \ln |u| + C \quad du = 8x \, dx\]
\[= \frac{1}{8} \ln |3 + 4x^2| + C.\]

\[F(0) = 0 \]
\[0 = \frac{1}{8} \ln |3| + C \quad \Rightarrow \quad C = - \frac{1}{8} \ln 3.\]

\[\therefore \quad F(x) = \frac{1}{8} \ln |3 + 4x^2| - \frac{1}{8} \ln 3.\]
\( u = \text{e}^{2t} \)  
\( du = 2\text{e}^{2t} \, dt \)

\[
\int \text{e}^{2u} \sin(u) \, \frac{du}{2} = \frac{1}{2} \left( -\cos(u) \right) \bigg|_1^\text{e}^{2u} \\
= -\frac{1}{2} \left[ \cos\left(\text{e}^{2u}\right) - \cos(1) \right] \text{e}^{2u} \\
= -\frac{1}{2} \left( \cos\left(\text{e}^{2u}\right) - \cos(1) \right) \text{e}^{2u} \\
= \frac{1}{2} \cdot x^2
\]