1. Compute the indicated derivatives of the function \( y = f(x) \). It is not necessary to simplify your answer.

(a) (4 points) \( f'(x) \) where

\[
f(x) = e^{1 - \cos(2x)}
\]

\[
f' = e^{1 - \cos(2x)} \cdot \left( -2 \sin(2x) \right)
\]

(b) \( f'(x) \) where

\[
y = f(x) = (x + 1)^x
\]

\[
\ln y = x \ln(x + 1)
\]

\[
\frac{y'}{y} = \frac{\ln(x + 1)}{x + 1} + x \frac{1}{x + 1}
\]

\[
f' = (x + 1)^x \left( \ln(x + 1) + \frac{x}{x + 1} \right)
\]

(c) \( f'(x) \) where

\[
f(x) = \arctan(2x - 3)
\]

\[
\frac{d}{dx} (2x - 3)
\]

\[
= \frac{2}{1 + (2x - 3)^2}
\]
(d) \[ f^{(3)}(x) \] where \[ f(x) = \ln(1 - x) \]

\[
\begin{align*}
    f' &= \frac{-1}{1-x} = (x-1)^{-1} \\
    f'' &= -(x-1)^{-2} \\
    f''' &= 2(x-1)^{-3}
\end{align*}
\]
2. Compute the limit

\[
\lim_{{x \to 0}} \frac{x^4}{\sin(x) \sin(2x) \sin(4x) \sin(8x)}
\]

Or

\[
\lim_{{x \to 0}} \frac{x - \sin x}{x(1 - \cos x)}
\]

\[
\lim_{{x \to 0}} \left( \frac{x}{\sin x} \right) \left( \frac{2x}{\sin 2x} - \frac{1}{2} \right) \left( \frac{4x}{\sin 4x} - \frac{1}{4} \right) \left( \frac{8x}{\sin 8x} - \frac{1}{8} \right)
\]

\[
= 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8}
\]

\[
= \frac{1}{64}
\]

\[
\lim_{{x \to 0}} \frac{x - \sin x}{x(1 - \cos x)}
\]

\[
\lim_{{x \to 0}} \frac{1 - \cos x}{(-\cos x) + x \sin x}
\]

\[
\lim_{{x \to 0}} \frac{\sin x}{\sin x + \sin x + x \cos x}
\]

\[
\lim_{{x \to 0}} \frac{\cos x}{\cos x + \cos x + \cos x + (-x \sin x)}
\]

\[
= \frac{1}{1 + 1 + 1 + 0} = \frac{1}{3}
\]
3. Let \( y = f(x) = x^4 - 2x^2 + 9 \)

with its natural domain. Determine where \( f(x) \) is increasing and where it is decreasing. Find all local maxima and minima. State the name of a test" in your reasoning.

Also find the interval where \( f \) is concave up and where \( f \) is concave down.

Is any of these local maxima or minima an absolute maximum or absolute minimum?

\[
f(x) = 4x^3 - 4x
\]

\[
= 4x(x^2 - 1) = 0
\]

\( x = 0, \pm 1 \) are critical numbers.

\[
\begin{array}{c|c|c|c|c}
 x & f'(x) & & & f''(x) \\
\hline
-1 & + & - & + & \\
0 & - & + & - & \\
1 & + & - & + & \\
\end{array}
\]

1) \( f \) is increasing on \((1, \infty), (-1, 0)\).

2) \( f \) is decreasing in \((-\infty, -1), (0, 1)\).

By first derivative test, \( f \) changes from negative to positive, \( f(-1), f(1) \) are local minimum values

\( f \) changes from positive to negative, then \( f(0) \) is a local maximum value.

3) \( f''(x) = 12x^2 - 4 \Rightarrow x = \pm \frac{1}{\sqrt{3}} \).

\( f'' > 0 \Rightarrow x > \frac{1}{\sqrt{3}} \) concave up.

\( f'' < 0 \Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \) concave down.

4) Absolute minimum values are \( f(1) \) and \( f(-1) \).
4. Let \( y \) be a function of \( x \) such that \( y^2 + 3y + x^3 = -1 \) and \( y = -2 \) when \( x = 1 \) and the derivatives \( y' \) and \( y'' \) exist at \( x = 1 \).

(a) \( \text{Compute } y' \text{ when } x = 1 \)

\[
2yy' + 3y + 3x^2 = 0.
\]
\[
y' = \frac{-3x^2}{2y+3}.
\]
\[
y' = \frac{-3}{-4+3} = 3.
\]

(b) \( \text{Compute } y'' \text{ when } x = 1 \)

\[
y' = \frac{-3x^2}{2y+3}.
\]
\[
y'' = \frac{-6x(2y+3) - 2y(-3x^2)}{(2y+3)^2} = \frac{-12xy - 18x + 6x^2 y'}{(2y+3)^2}.
\]
\[
y'' = \frac{-12xy - 18x + 6x^2 \left( \frac{-3x^2}{2y+3} \right)}{(2y+3)^2}.
\]
5. Verify that the function $f(x) = -x^3 + 5$ satisfies the hypotheses of the Mean Value Theorem on the interval $[-1, 0]$. Then find all numbers $c$ that satisfy the conclusion of the Mean Value Theorem.

1. $f(x)$ is polynomial, so $f$ is continuous on $[-1, 0]$ and differentiable on $(-1, 0)$.

2. By Mean Value Theorem,

$$f(0) - f(-1) = f'(c) (0 - (-1))$$

$$5 - 6 = f'(c)$$

$$f'(c) = -1$$

$$f'(x) = -3x^2$$, so

$$-3c^2 = -1$$

$$c^2 = \frac{1}{3}$$

$$c = \pm \frac{1}{\sqrt{3}}$$ (since $\sqrt{3}$ is not in interval $(-1, 0)$)

So, $c = -\frac{1}{\sqrt{3}}$. **}$
6.

Spherical Volume. The volume $V$ of a spherical cancer tumor is given by $V = \frac{\pi x^3}{6}$, where $x$ is the diameter of the tumor. A physician estimates that the diameter is growing at the rate of $4$ millimeters per day, at a time when the diameter is already 10 millimeters. How fast is the volume of the tumor changing at that time? (Simplify your answer)

\[
\frac{dx}{dt} = 0.4
\]

\[
\frac{dV}{dt} = \frac{\pi}{6} 3x^2 \frac{dx}{dt}
\]

\[
= \frac{\pi}{2} 100 (0.4)
\]

\[
= 20\pi \text{ millimeter/day}
\]
A soccer ball is made of leather \( \frac{1}{8} \) inch thick and its inner diameter is 9 inches. Use differentials to estimate the volume of its leather shell.

\[ r \text{ : radius of the ball} \quad r = 4.5 \text{ inch} \]

\[ V = \text{volume} \quad V = \frac{4}{3} \pi r^3 \]

\[ dv = 4\pi r^2 \, dr \]

\[ = 4\pi \left(4.5\right)^2 \frac{1}{8} \]

\[ = 4\pi \frac{81}{4} \cdot \frac{1}{8} = \frac{81}{8} \pi \quad \text{(in}^3) \]

Use differentials to estimate \( \sqrt[3]{26} \).

\[ f(x) = \sqrt[3]{x} \quad f'(x) = \frac{1}{3} x^{-\frac{2}{3}} \]

\[ f(26) - f(27) \approx f'(27)(26 - 27) \]

\[ f(26) \approx f(27) + f'(27)(-1) \]

\[ \approx 3 + \left(-\frac{1}{27}\right) = \frac{80}{27} \]

\[ x \]