Quick Reivew from Last lecture

- Standard Basic vectors: in 3D,

$$\mathbf{i} = \langle 1, 0, 0 \rangle$$
$$\mathbf{j} = \langle 0, 1, 0 \rangle$$
$$\mathbf{k} = \langle 0, 0, 1 \rangle$$

- (Inner product) Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

  Also, $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$, where θ is the angle between vectors $\vec{a}$ and $\vec{b}$.

- (Cross product) Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

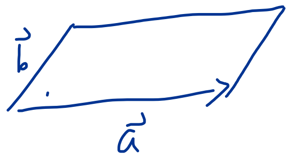
$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| |\sin(\theta)|.$$

- (Parametric equations of a line) In 3D, a line through point (x_0, y_0, z_0), in direction of $\langle a, b, c \rangle$

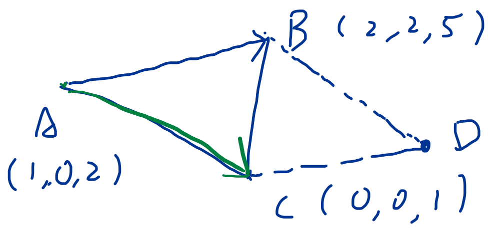
$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

Example: Find the area of the triangle with vectices $\overset{A}{(1,0,2)}$, $\overset{B}{(2,2,5)}$, $\overset{C}{(0,0,1)}$.

Recall:



area of parallelogram $= \|\vec{a} \times \vec{b}\|$.



$\vec{AB} = B - A$

$= (2,2,5) - (1,0,2)$

$= (1, 2, 3)$

$\vec{AC} = C - A$

$= (0,0,1) - (1,0,2) = (-1,0,-1)$

$$\text{area of Triangle} = \frac{\|\vec{AB} \times \vec{AC}\|}{2} = \frac{1}{2}\|(-2,-2,2)\| = \sqrt{3}. \ \#$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} i - \underbrace{\begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix}}_{\substack{\| \\ -1+3}} j + \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} k$$

$= -2i - 2j + 2k.$

$= \langle -2, -2, 2 \rangle.$

§**Geometry of determinants** We can discover a link between 2×2 determinants and area, and a link between 3×3 determinants and volume.

- 2×2 determinants: the absolute value of 2×2 determinants $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ is the area of the parallelogram spanned by vectors $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\vec{b} = b_1\mathbf{i} + b_2\mathbf{j}$.

area of the parallelogram

$= \|\vec{a}\times\vec{b}\|$.

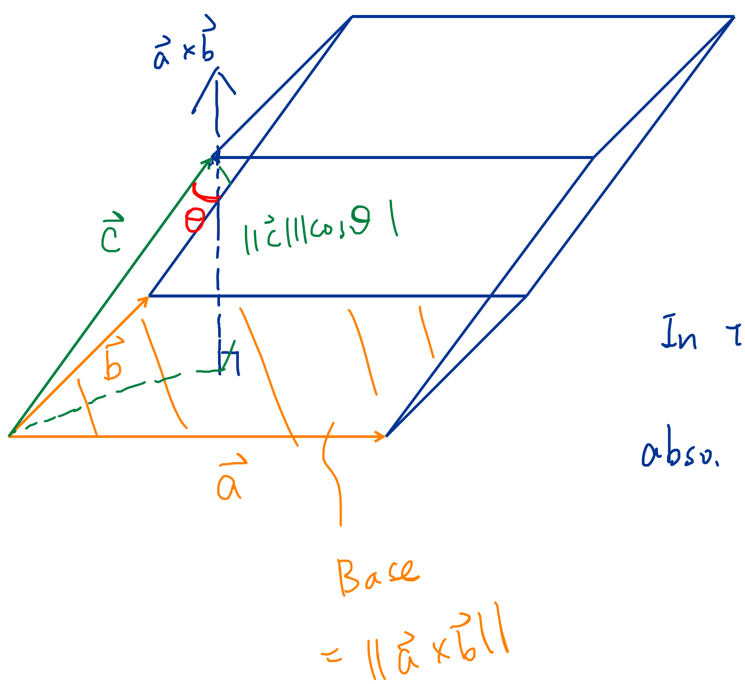
$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_2 & 0 \\ b_2 & 0 \end{vmatrix} i - \begin{vmatrix} a_1 & 0 \\ b_1 & 0 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k = \left\langle 0, 0, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle,$$

- 3×3 determinants: the absolute value of $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is the volume of the parallelepiped spanned by vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{c} = \langle c_1, c_2, c_3 \rangle$



Q: Why abso. value of $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is the volume of the parallelepiped on the left ?

In text, page 36, 40, we get

$$\text{abso. value of } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = |(\vec{a}\times\vec{b})\cdot\vec{c}|$$

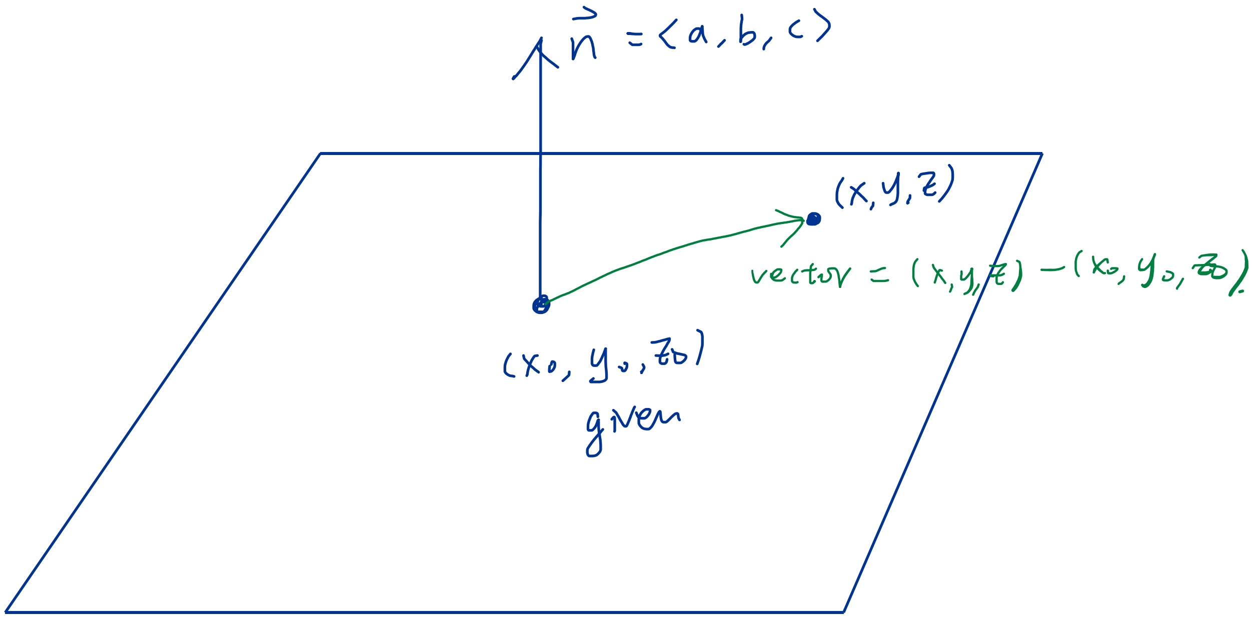
$$= \underbrace{\|\vec{a}\times\vec{b}\|}_{\text{Base area}}\underbrace{\|\vec{c}\||\cos\theta|}_{\text{height.}}$$

θ: angle between $\vec{a}\times\vec{b}$ and $\vec{c}$.

§Equations of Planes

Ingredients:

Need ___point___ in the plane
and ___a normal vector___ for the plane.
$\langle a, b, c \rangle$.



Plane

$$0 = \vec{n} \cdot \big((x, y, z) - (x_0, y_0, z_0) \big)$$

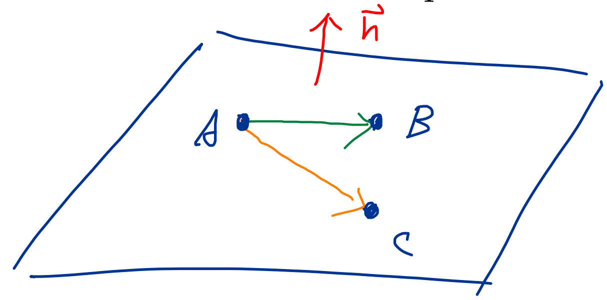
$$0 = (a, b, c) \cdot (x - x_0, \; y - y_0, \; z - z_0)$$

$$\boxed{0 = a(x - x_0) + b(y - y_0) + c(z - z_0)}$$

Equation of a plane.

Example: Find the equation of the plane that contains the three points $(0,1,3)$, $(1,1,0)$, $(3,0,-1)$.

$\quad A \qquad\quad B \qquad\quad C$



Choose $A = (0, 1, 3)$,

To find a normal vector:

$\overrightarrow{AB} = B - A = (1, 0, -3)$

$\overrightarrow{AC} = C - A = (3, -1, -4)$

They are $\perp$ to the plane.

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

$$= \begin{vmatrix} i & j & k \\ 1 & 0 & -3 \\ 3 & -1 & -4 \end{vmatrix} = \langle -3, -5, -1 \rangle.$$

Equation of the plane is

$$0 = (-3, -5, -1) \cdot (x - 0, y - 1, z - 3)$$

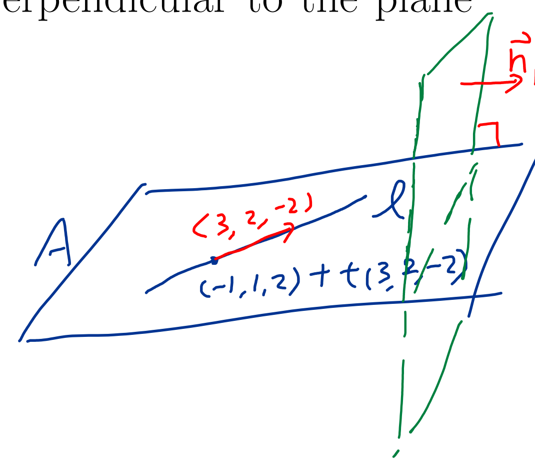
$$\Rightarrow 3x + 5y + z - 8 = 0 .$$

#

Example: (similar to ♯ 35 in Textbook) Find the equation of the plane that contains the line $l(t) = (-1, 1, 2) + t(3, 2, -2)$ and is perpendicular to the plane $\boxed{x + 2z = 7.}$ → normal $\vec{n}_1 = (1, 0, 2)$

point $\boxed{(-1, 1, 2)}$.

$(3, 2, -2) \parallel$ the plane A



To find normal vector of A.

$\vec{n} = (3, 2, -2) \times (1, 0, 2)$

$(4, -8, -2)$.

The equation of the plane

$0 = (4, -8, -2) \cdot (x+1, y-1, z-2)$

$= 4(x+1) + (-8)(y-1) - 2(z-2)$

$= 4x - 8y - 2z + 16$

$0 = 4x - 8y - 2z + 16$

$0 = 2x - 4y - z + 8$. #