§Lnear Transformation.

In general, given any $m \times n$ matrix B, we can define a function $g : \mathbb{R}^n \to \mathbb{R}^m$ by $g(\mathbf{x}) = B\mathbf{x}$ that maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then g is a linear transformation.

On the other hand, given any function g, then g may not be a linear transformation.

For example, $g(x, y) = (x^2, y, x)$ and $g(x, y, z) = (x, xy)$ are not linear transformations.

Example 6. *For $f(x, y) = (4x + 2y,\ y/\pi, x + y)$, can you find a matrix A such that $f(\boldsymbol{x}) = A\boldsymbol{x}$.*

§Determinants and linear transformations. A **linear transformation** $T : \mathbb{R}^n \to \mathbb{R}^n$, then T is associated with a $n \times n$ matrix.

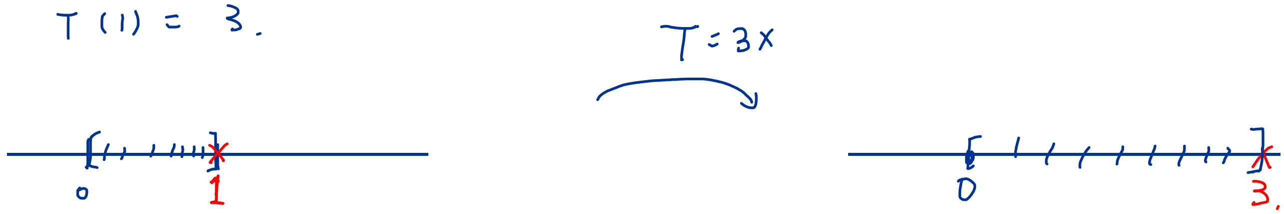
1. A **linear transformation** $T : \mathbb{R}^1 \to \mathbb{R}^1$ of the form $T(x) = ax$ for some scalar a.

$= [a][x] = [ax].$

Example 7. *(a) A one-dimensional linear transformation* $T(x) = 3x$.

$T(0) = 0$

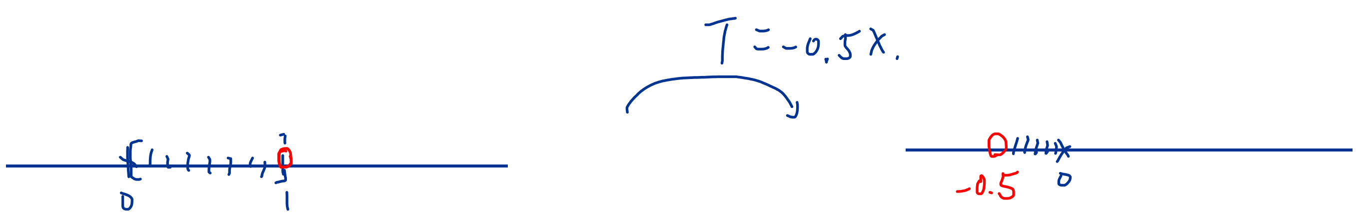
$T(1) = 3.$



T maps $[0, 1]$ onto $[0, 3]$.

the length is increased by a factor of $\boxed{3}$

(b) $T(x) = -0.5x$.



① length of $[-0.5, 0]$ is decreased by a factor of $|-0.5|$.

② sign of (-0.5) implies that T reverses the orientation. #

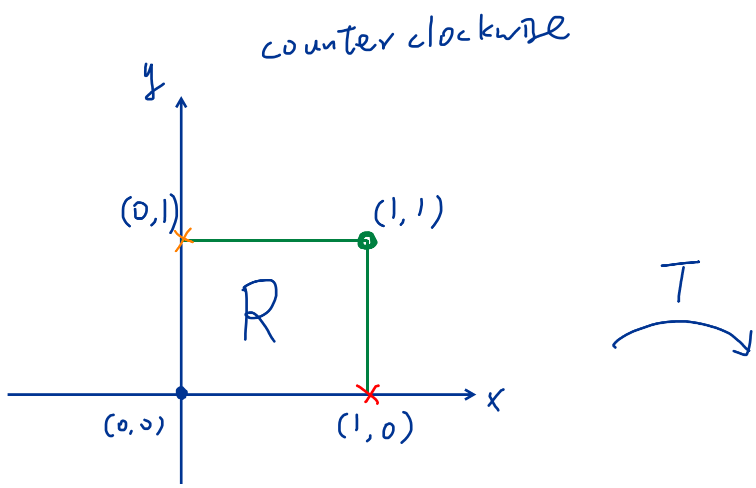
2. A **linear transformation** $T : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

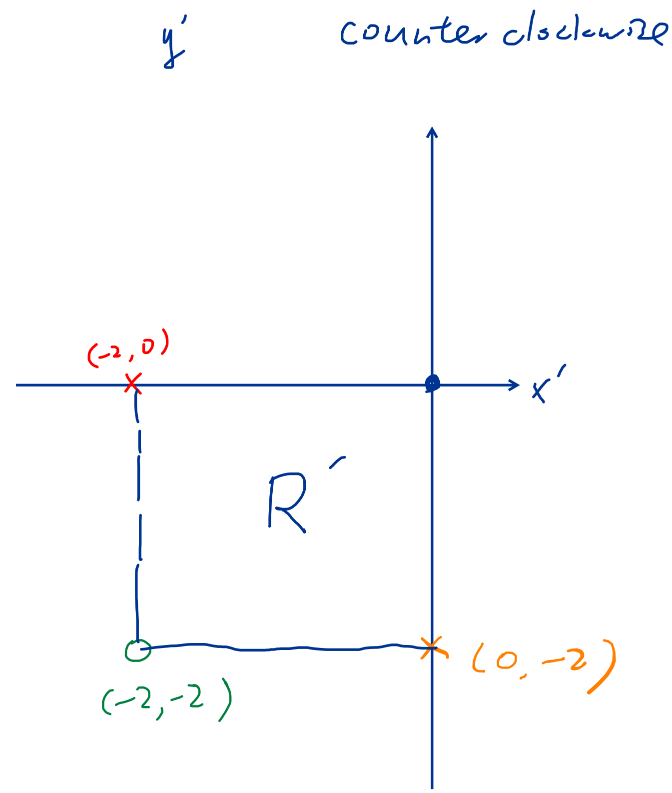
$$T(x, y) = (ax + by,\ cx + dy) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \begin{bmatrix} x \\ y \end{bmatrix}.$$

where a, b, c, d are numbers.

Example 8. *(a) A two-dimensional linear transformation*

$$T(x, y) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$





$T(0,0) = (0,0)$

$T(1,0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$T(1,1) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$

$T(0,1) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$

area $R = 1$

area $R' = 4.$

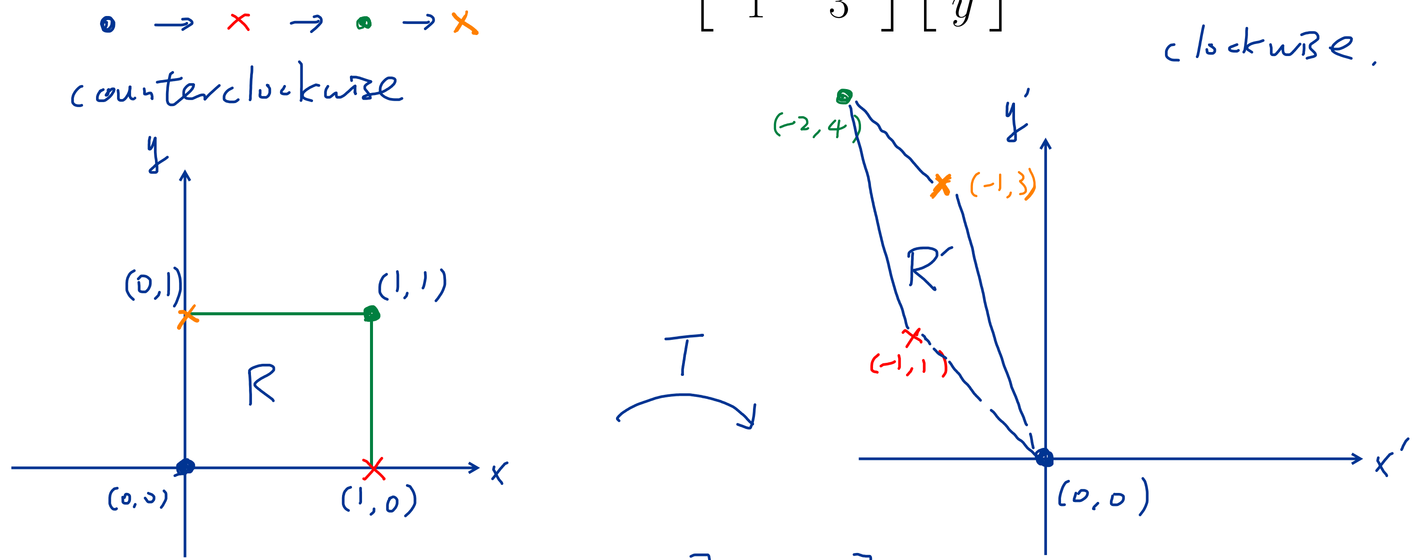
① area $R' = \boxed{4}$ area R

$\left| \det \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \right|$

② Since $\det \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} > 0$, T preserves the orientation.

(b) A two-dimensional linear transformation

$$T(x,y) = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$



$$T(1,0) = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$T(1,1) = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$T(0,1) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

① area of R' will increase by a factor of $\left|\det\begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}\right| = 2$.

② $\det\begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix} < 0$, so T reverses the orientation.

3. A **linear transformation** $T : \mathbb{R}^3 \to \mathbb{R}^3$ of the form

$$T(\mathbf{x}) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ -3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$\det \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ -3 & -1 & 2 \end{bmatrix} = 12.$$

① $\det [\ ] > 0$, it preserves the orientation.

② T expands the volumes of objects by a factor of 12.

③ T maps "parallelepiped" into "parallelepiped".

⌈ See math insights (Part 3) for pictures ⌋. (online reading

§How linear transformations map parallelograms and parallelepipeds?

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\det(A) \neq 0$. A 2-dimensional linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(x, y) = (ax + by,\ cx + dy) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where a, b, c, d are numbers. Then T maps **parallelograms** onto **parallelograms** and vertices into vertices.

If A is a 3×3 matrix with $\det(A) \neq 0$, then a 3-dimensional linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\mathbf{x} = A\mathbf{x}$ maps **parallelepipeds** onto **parallelepipeds**.

§Geometric properties of the determinant.

We have learned that the determinant of a square matrix can be related to the **area or volume of a region**.

In particular, for the linear transformation $f(\mathbf{x}) = A\mathbf{x}$, the determinant of A reflects how the linear transformation f can **scale or reflect objects.**

1. The absolute value of the determinant reflects how the linear transformation T **expands or compresses objects**.

Properties:

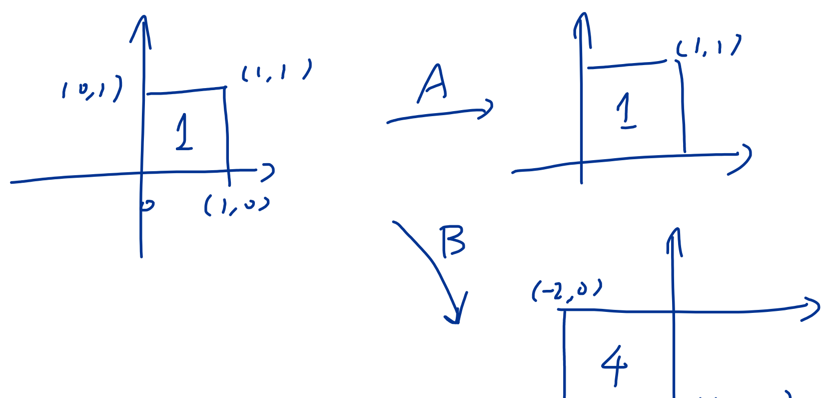
- $|\det(cA)| = c^n|\det(A)|$ in n-dimensions.

Example 9. To see $|\det(B)| = 2^2 |\det A|$.

$B = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$B = 2A$



2. The sign of the determinant determines whether the linear transformation T **preserves or reverses orientation**.

- $\det(cA) = c^n\det(A)$ in n-dimensions.

3. The effect of multiplying matrices.

- $\det(AB) = \det(A)\det(B)$.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

4. The determinant of a matrix inverse.

- $\det(A^{-1}) = \frac{1}{\det(A)}$. Note that here $\det(A)$ is not zero.

A^{-1}, inverse of A, satisfies $A^{-1}A = AA^{-1} = I_n$,

$\det A \neq 0$, we call A is invertible. #