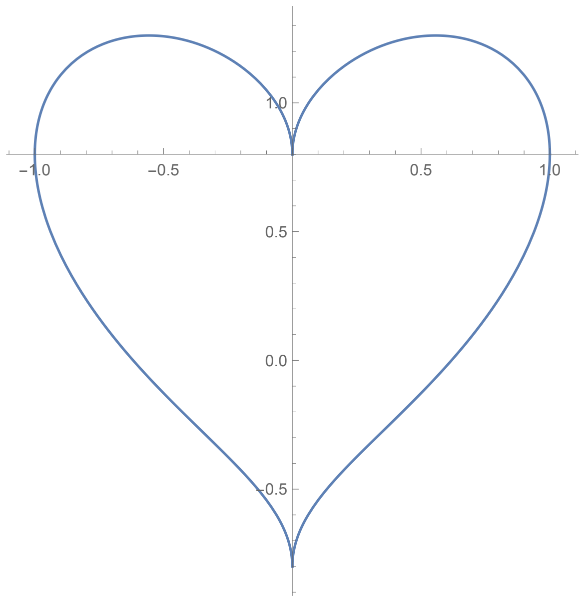
Math 2374 Spring 2018 - Week 4

Quick Reivew from previous lecture

- The velocity of the path $c(t)$ is $c'(t)$. The speed of the path is $\|c'(t)\|$.
- Tangent line to a path at point $c(t_0)$ is
$$l(t) = c(t_0) + (t - t_0)c'(t_0).$$
- $\mathbf{D}(cf)(x_0) = c\mathbf{D}f(x_0)$
- $\mathbf{D}(f + g)(x_0) = \mathbf{D}f(x_0) + \mathbf{D}g(x_0)$
- $\mathbf{D}(fg)(x_0) = g(x_0)\mathbf{D}f(x_0) + f(x_0)\mathbf{D}g(x_0)$
- $\mathbf{D}\left(\frac{f}{g}\right)(x_0) = \frac{g(x_0)\mathbf{D}f(x_0) - f(x_0)\mathbf{D}g(x_0)}{[g(x_0)]^2}$
- $\mathbf{D}(f \circ g)(x_0) = \mathbf{D}f(g(x_0))\ \mathbf{D}g(x_0)$
  - Example:
    Suppose $c(t) = (x(t), y(t), z(t))$ is a path and $f : \mathbb{R}^3 \to \mathbb{R}^1$. Then
$$\mathbf{D}(f \circ c)(t) = \mathbf{D}f(c(t))\mathbf{D}c(t)$$
    or
$$\mathbf{D}(f \circ c)(t) = \nabla f(c(t)) \cdot c'(t)$$

How do you draw a picture like this in Mathematica?



The curve[1] is

$$x^2 + (\frac{5y}{4} - \sqrt{|x|})^2 = 1.$$

[1]This picture is created by Mathematica. It consists of 4 different paths.

Example 3. *The trajectory of a bird is given by the path*

$$c(t) = (t^2,\ \tan(t),\ t^4 + 7t).$$

bird's position at time t.

The temperature at each point of the space is measured by a function $f(\overbrace{x, y, z}^{\text{position}}) = \underbrace{xy^2 + z}_{\text{temp. at } (x,y,z).}$*. Find the rate of change of temperature that the bird is experiencing at any given time* $t = 0$.

The temperature at time t the bird is feeling is $(f \circ c)(t) = \underbrace{f(\underbrace{c(t)}_{\text{position}})}_{\text{temperature at position } c(t).}$

$$D(f \circ c)(t) \overset{\text{chain rule}}{=} Df(c(t))\, Dc(t)$$

$$= \begin{bmatrix} y^2 & 2xy & 1 \end{bmatrix}_{1\times 3}\Big|_{(x,y,z)=c(t)} \begin{bmatrix} 2t \\ \sec^2 t \\ 4t^3 + 7 \end{bmatrix}_{3\times 1}$$

$$= \begin{bmatrix} \tan^2 t & 2t^2 \tan t & 1 \end{bmatrix} \begin{bmatrix} 2t \\ \sec^2 t \\ 4t^3 + 7 \end{bmatrix}$$

$$D(f \circ c)(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix}$$

$$= 7. \ \#$$

✓ Orienting curves

For example, given a curve C (parabola) from point $p = (-1, 1)$ to $q = (2, 4)$. Let C be parametrized by the function

$$c(t) = (t, t^2)$$

for $-1 \leq t \leq 2$. It has unit tangent vector

$$T = \frac{c'(t)}{\|c'(t)\|} = \frac{\langle 1, 2t\rangle}{\sqrt{1+4t^2}}$$

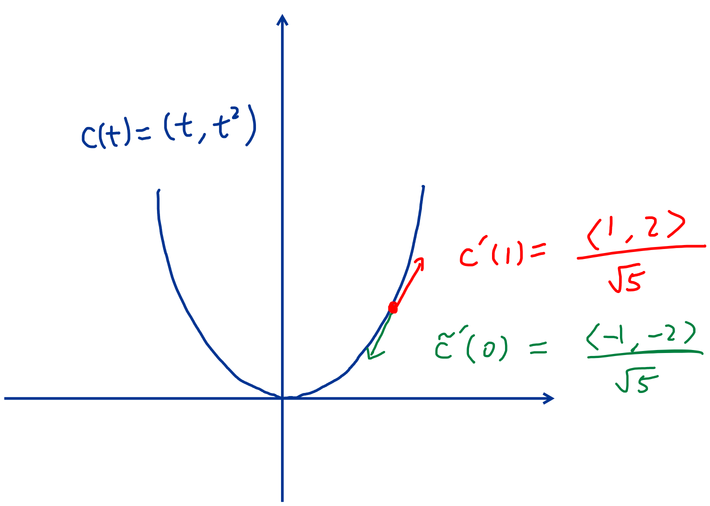
We could also parametrize the curve C "backward", that is, going from q to p. Thus,

$$\tilde{c}(s) = (1 - s, (1-s)^2)$$

for $-1 \leq s \leq 2$.

Its unit tangent vector is

$$T = \frac{\tilde{c}'(s)}{\|\tilde{c}'(s)\|} = \frac{\langle -1,\ -2(1-s)\rangle}{\sqrt{1+4(1-s)^2}}$$

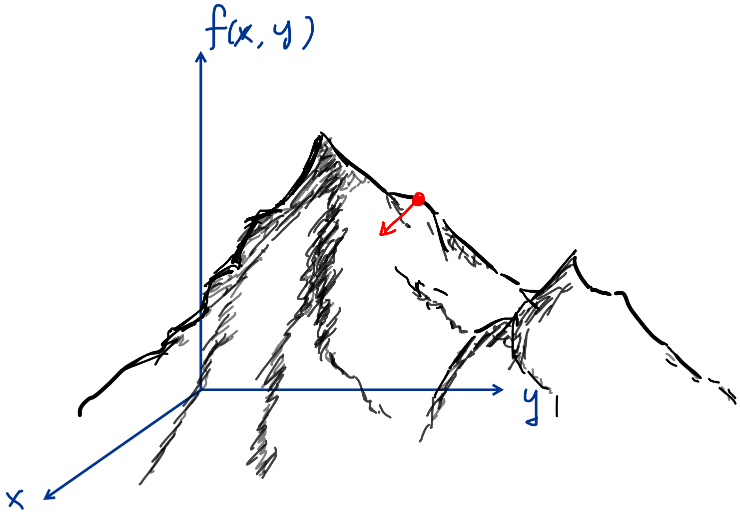


NOTE: $c(1) = \tilde{c}(0)$ the same point.

2.6 Gradients and Directional Derivatives

Motivation:

Let the function $f(x, y)$ be the height of a mountain at each point $\mathbf{x} = (x, y)$. Suppose you are standing at point $\mathbf{x} = \mathbf{a}$. Then the *slope of the ground* in front of you will depend on the *direction you are facing.*



Recall that the partial derivatives of f will give:

- the slope $\frac{\partial f}{\partial x}$ in the positive x direction;

- the slope $\frac{\partial f}{\partial y}$ in the positive y direction.

To generalize the partial derivatives to calculate the slope in any direction. This is called the **Directional Derivatives**.

We formalize this concept as follows:

If $f : \mathbb{R}^n \to \mathbb{R}^1$, the **directional derivative** of f at the point $\mathbf{a}$ in the direction $\mathbf{v}$ is

$$\mathbf{D}_{\mathbf{v}} f(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{v}) - f(\mathbf{a})}{h}$$

Think: $\begin{cases} a & \text{position} \\ v: & \text{travel direction.} \end{cases}$

if this limit exists. Note that we usually let $\mathbf{v}$ to be a **unit** vector!

Ex: vector $w = \langle 1, 2, 3 \rangle$.

unit vector $\frac{w}{\|w\|} = \frac{\langle 1, 2, 3 \rangle}{{}^{3}\sqrt{14}}$. #

✓ *Note that* $\boldsymbol{D}_{\boldsymbol{v}}f(\boldsymbol{a})$ *is a number, not a matrix.* $\boldsymbol{D}_{\boldsymbol{v}}f(\boldsymbol{a})$ *is the slope of* $f(x,y)$ *when standing at the point* $\boldsymbol{a}$ *and facing the direction* $\boldsymbol{v}$*.*

We denote

$$\mathbf{D}_{\mathbf{v}}f(x) = \nabla f(x) \cdot \mathbf{v}$$

Recall that **Gradients in** $\mathbb{R}^3$

If $f : \mathbb{R}^3 \to \mathbb{R}^1$, the **gradient of** f,

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

We also denote it by $\mathrm{grad} f$.

Example 1. $f(x, y, z) = x^2 + xy + 3z$. *Find the rate of change for f in the direction $w = \langle 1, 2, 1\rangle$ at point $(1, 0, 0)$.*

Note that it is the same as asking the directional derivative of f at point $(1, 0, 0)$ along the vector w.

$$D_w f(x,y,z) = \nabla f(x,y,z) \cdot \frac{w}{\|w\|}.$$

$$\nabla f = \langle 2x + y,\ x,\ 3\rangle. \Rightarrow \nabla f(1,0,0) = \langle 2, 1, 3\rangle.$$

$$\frac{w}{\|w\|} = \frac{\langle 1, 2, 1\rangle}{\sqrt{1+4+1}} = \frac{\langle 1, 2, 1\rangle}{\sqrt{6}}.$$

$$D_w f(1,0,0) = \langle 2, 1, 3\rangle \cdot \frac{\langle 1, 2, 1\rangle}{\sqrt{6}}$$

$$= \frac{7}{\sqrt{6}} = \frac{7\sqrt{6}}{6}. \#$$

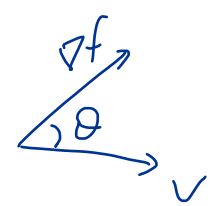
Fact. *If $\nabla f(x) \neq 0$, then $\nabla f(x)$ points in the direction of the fastest increase of f.*

Explanation: We know the rate of change of f in direction v (unit vector) is

$$D_v f = \nabla f \cdot v.$$

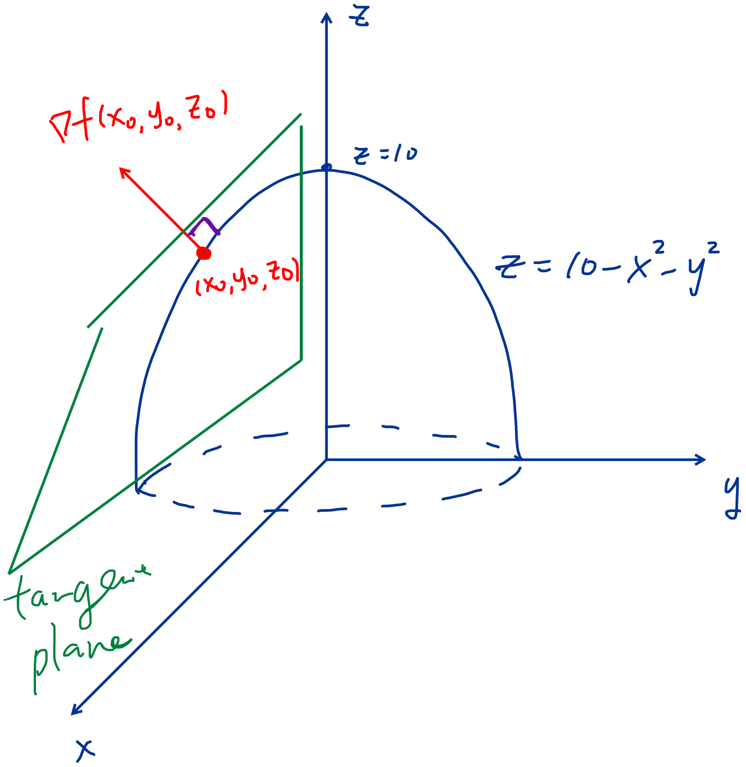
$$= \|\nabla f\| \|v\| \cos\theta$$

$$= \|\nabla f\| \; 1 \; \cos\theta.$$



The maximum value of $D_v f$ is $\|\nabla f\|$ when $\theta = 0$.

Fact. *If $f : \mathbb{R}^3 \to \mathbb{R}^1$ has continuous partial derivatives. Suppose that (x_0, y_0, z_0) lie on the level surface $f(x, y, z) = c$ for some constant c. Then $\nabla f(x_0, y_0, z_0)$ is normal to the level surface.*



EX: $f(x,y,z) = z + x^2 + y^2$.

Consider level surface (set) $f(x,y,z) = 10$. Then

$$z = 10 - x^2 - y^2.$$

§Tangent planes to level surfaces

The tangent plane of the level surface $f(x, y, z) = c$ (c is constant) at point (x_0, y_0, z_0) is

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

if $\nabla f(x_0, y_0, z_0) \neq 0$.

✓**Note that $\nabla f(x_0, y_0, z_0)$ is the "normal vector" to the tangent plane of the surface "$f =$ constant"** at (x_0, y_0, z_0).

Example 2. *Find the equation of the plane tangent to the surface*

$$3xy + 10e^{-y^2} = -z^2y + \cancel{e^{100}} \ 10$$

at the point $(2, 0, 1)$.

Consider $f(x,y,z) = 3xy + 10e^{-y^2} + z^2y$.

$f(x,y,z) = 10$.

$\nabla f = \langle 3y, \ 3x - 20ye^{-y^2} + z^2, \ 2zy \rangle$.

$\nabla f(2,0,1) = \langle 0, \ 6 - 0 + 1, \ 0 \rangle$

$= \langle 0, 7, 0 \rangle$ normal vector

Tangent plane is

$\nabla f(2,0,1) \cdot (x-2, y-0, z-1) = 0$

$\langle 0, 7, 0 \rangle \cdot (x-2, y, z-1) = 0$

$\Rightarrow \quad \underline{y = 0}$ #

Example 3. *Let* $A(\overbrace{x, y, z}^{\text{position}}) = \overbrace{1 - x^2 - e^y z}^{\text{pressure.}}$ *is the atmospheric pressure at position* (x, y, z)*. If you were at position* $(2, 0, 1)$*, find the direction that you would need to move in order to* $\boxed{\underline{\textit{decrease}}}$ *the atmospheric pressure asap. Write the answer in the form of a* <u>*unit vector*</u>*.*

$$\nabla A = \langle -2x, -e^y z, -e^y \rangle.$$

$$\nabla A(2, 0, 1) = \langle -4, -1, -1 \rangle.$$

$$-\nabla A(2, 0, 1) = \langle 4, 1, 1 \rangle.$$

$$\frac{\langle 4, 1, 1 \rangle}{\| \langle 4, 1, 1 \rangle \|} = \frac{\langle 4, 1, 1 \rangle}{\sqrt{18}}$$

$$= \frac{\langle 4, 1, 1 \rangle}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \langle 4, 1, 1 \rangle \cdot \frac{\sqrt{2}}{6}.$$

$$= \left\langle \frac{2}{3}\sqrt{2}, \frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6} \right\rangle. \quad \#$$