Math 2374 Spring 2018 - Week 6

(1) Triple integral

$$\int\int\int_W f(x,y,z)dV.$$

In particular, if $f=1$, then $\int\int\int_W f(x,y,z)dV$ is the volume of the region W.

(2) **idea: reduce a "triple" integral into a "double" integral.**

Shadow method:

Imagine a sun is on z axes.

$$\int\int\int_W f(x,y,z)dV = \int\int_{shadow}\left(\int_{bottom(x,y)}^{top(x,y)} f(x,y,z)dz\right)dxdy.$$

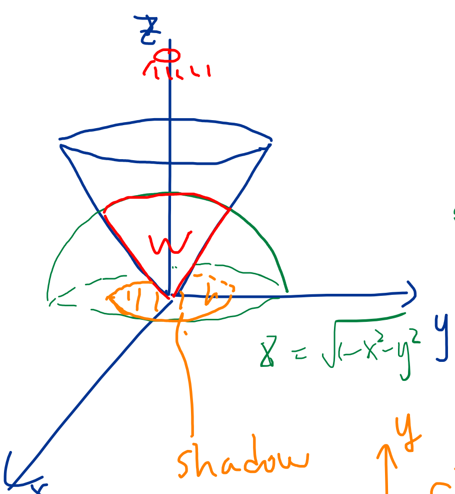
EX: W is bounded by cone $z=\sqrt{x^2+y^2}$ and upper sphere $z=\sqrt{1-x^2-y^2}$.

Find volume of W.

Find intersection.

$\sqrt{x^2+y^2}=\sqrt{1-x^2-y^2}$.

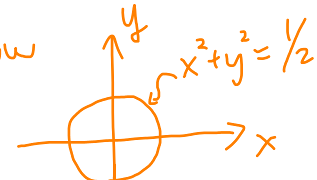
$x^2+y^2=\frac{1}{2}$.



$\sqrt{x^2+y^2}\le z\le\sqrt{1-x^2-y^2}$

shadow
$-\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$
$-\sqrt{\frac{1}{2}-x^2}\le y\le\sqrt{\frac{1}{2}-x^2}$

$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 1 \quad dz\,dy\,dx$$



Example 3. *Evaluate*

$$\int\int\int_R x\,dV$$

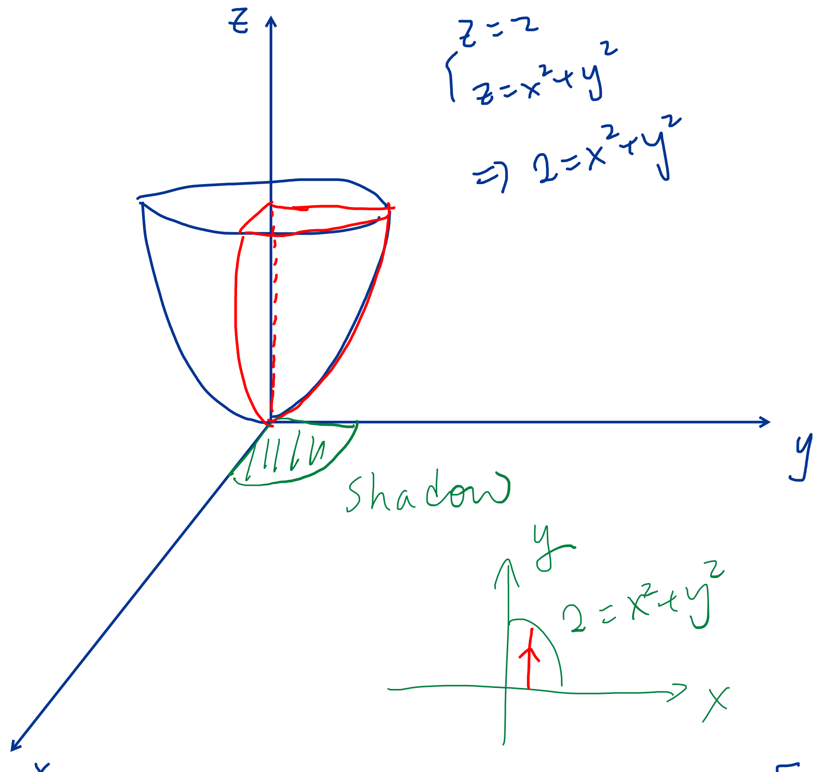
with R is the region enclosed by the planes $x=0$, $y=0$, and $z=2$ and the surface $z=x^2+y^2$ and lying in the quadrant $x\geq 0$, $y\geq 0$.

> *Note that to plot the paraboloid $z=x^2+y^2$. One way is using the following code in Mathematica (see Lab 07 file):
>
> ```
> f[r_, theta_] = {r Cos[theta], r Sin[theta], r^2}
> ParametricPlot3D[f[r, theta], {r, 0, 1}, {theta, 0, 2 Pi}]
> ```

[Method 1]

intersection.

$$\begin{cases} z=2 \\ z=x^2+y^2 \end{cases} \Rightarrow 2=x^2+y^2$$



$$0 \leq x \leq \sqrt{2}$$

$$0 \leq y \leq \sqrt{2-x^2}$$

$$x^2+y^2 \leq z \leq 2$$

$$\int_0^{\sqrt2}\int_0^{\sqrt{2-x^2}}\int_{x^2+y^2}^{2} x \; dz\,dy\,dx = \int_0^{\sqrt2}\int_0^{\sqrt{2-x^2}} x(2-x^2-y^2)\;dy\,dx$$

$$= \int_0^{\sqrt2} 2xy - x^3y - \frac{x}{3}y^3 \Big|_0^{\sqrt{2-x^2}} dx.$$

$$= \int_0^{\sqrt2} 2x\sqrt{2-x^2} - x^3\sqrt{2-x^2} - \frac{x}{3}(2-x^2)^{3/2} dx$$

$u=2-x^2$

$du=-2x\,dx$

$x^2=2-u$

$$= -\frac{2}{3}(2-x^2)^{\frac32}\Big|_0^{\sqrt2} - \int_0^{\sqrt2} x^3\sqrt{2-x^2}\,dx + \frac{1}{3}\frac{1}{5}(2-x^2)^{\frac52}$$

$$= 7 \quad \sim \qquad -\int(2-u)\sqrt{u}\left(-\frac{du}{2}\right) + \sim$$

$$= 8\sqrt2/15. \quad \#$$

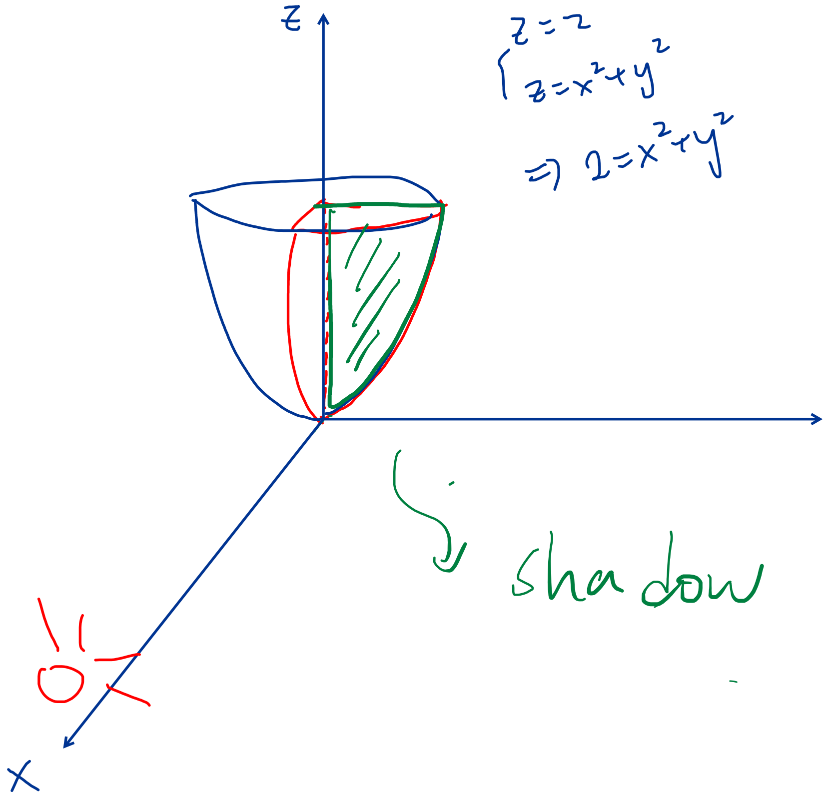
[Method 2]

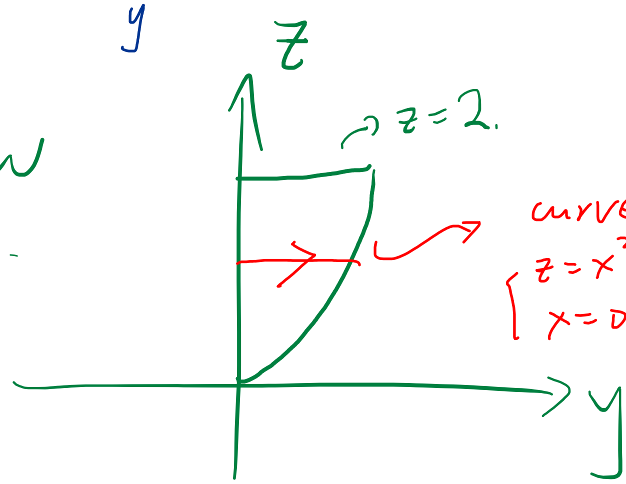
$$\begin{cases} z=2 \\ z=x^2+y^2 \end{cases} \Rightarrow 2=x^2+y^2$$

$$z = x^2+y^2$$

$$x = \pm\sqrt{z-y^2}$$

Take $x = \sqrt{z-y^2}$





curve $\begin{cases} z=x^2+y^2 \\ x=0 \end{cases}$ $z=y^2$.

$$0 \le x \le \sqrt{z-y^2}.$$

$$0 \le y \le \sqrt{z}.$$

$$0 \le z \le 2$$

$$\int_0^2 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-y^2}} x \, dx\, dy\, dz.$$

Example 4. *Let region W be bounded by the elliptic paraboloids $y = 5 - 4x^2 - z^2$ and $y = x^2 + z^2/4$. Set up the integral*

$$\int\int\int_W f(x, y, z)\,dV$$

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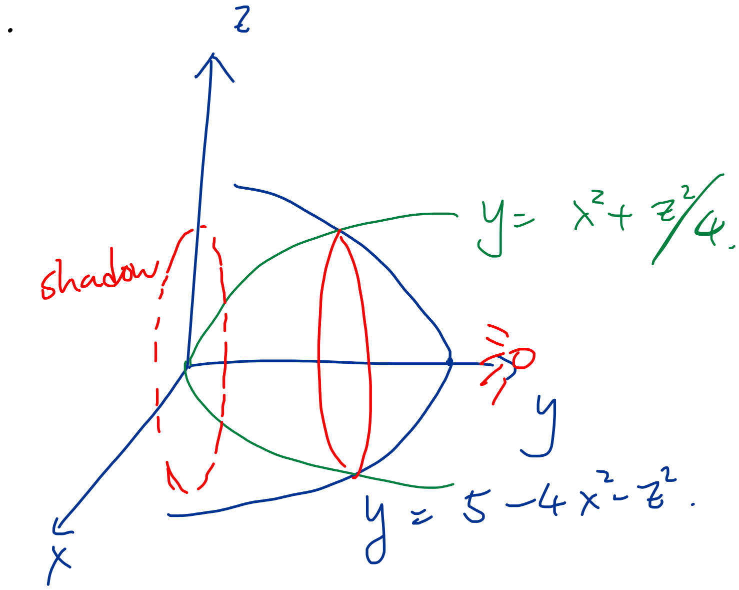


$$x^2 + z^2/4 \le y \le 5 - 4x^2 - z^2.$$

shadow $\begin{cases} -1 \le x \le 1 \\ -\sqrt{4-4x^2} \le z \le \sqrt{4-4x^2}. \end{cases}$

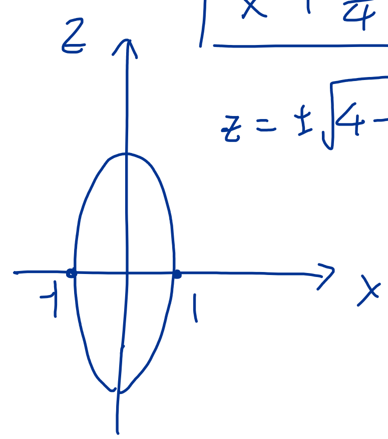
$$\int_{-1}^{1}\int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}}\int_{x^2+z^2/4}^{5-4x^2-z^2} f(x,y,z)\,dy\,dz\,dx.$$

※

intersection: $x^2 + z^2/4 = 5 - 4x^2 - z^2$

$\boxed{x^2 + \frac{z^2}{4} = 1.}$ ellipse

$z = \pm\sqrt{4-4x^2}$



4.1, 4.2 Acceleration, Newton's Second Law, and Arc Length.

Recall:

- $c(t) : \mathbb{R} \to \mathbb{R}^n$ is a parametrization of a curve. Suppose that
$$c(t) = (x_1(t), x_2(t), \cdots, x_n(t)).$$
Then
$$\mathbf{D}c = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}, \quad n \times 1 \text{ matrix.}$$
matrix of partial derivatives
- It can be written in vector form
$$c'(t) = (x_1'(t), x_2'(t), \cdots, x_n'(t)).$$
- $c'(t)$ is velocity.
- $\|c'(t)\|$ is speed.

Remark: How to find a parameterization for the straight-line path from the point (1,2,3) to the point (3,1,2).

$(1,2,3) \longrightarrow (3,1,2)$

(1,2,3) starting point

(3,1,2) end point.

$c(t) = (1,2,3) + t\left((3,1,2) - (1,2,3)\right)$

$0 \le t \le 1.$

$c(0) = (1, 2, 3)$

$c(1) = (3, 1, 2).$

§Velocity, acceleration

- $v(t) = c'(t)$ is the **velocity** of the curve.
- $a(t) = v'(t) = c''(t)$ is the **acceleration** of the curve.

Example 1. *If* $a(t) = \langle 4t,\ 7\sin(t),\ 3\rangle$ *with* $v(0) = \langle 3, 1, 0\rangle$ *and* $c(0) = \langle 1, 1, 0\rangle$. *Find* $c(t)$. Find $v(t)$, antiderivative of $a(t)$.

$$v(t) = \langle 2t^2 + C_1,\ -7\cos t + C_2,\ 3t + C_3\rangle.$$

$$\underset{(3,1,0)}{\overset{\shortparallel}{v(0)}} = (C_1,\ -7 + C_2,\ C_3) \Rightarrow C_1 = 3,\ C_2 = 8,\ C_3 = 0.$$

$$v(t) = (2t^2+3,\ -7\cos t + 8,\ 3t)$$

Find $c(t)$.

$$c(t) = \left(\tfrac{2}{3}t^3 + 3t + d_1,\ -7\sin t + 8t + d_1,\ \tfrac{3}{2}t^2 + d_3\right\rangle$$

$$\underset{(1,1,0)}{\overset{\shortparallel}{c(0)}} = (d_1,\ d_2,\ d_3) \Rightarrow d_1 = 1,\ d_2 = 1,\ d_3 = 0.$$

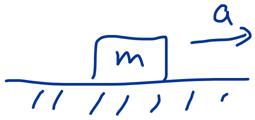
$$c(t) = \left(\tfrac{2}{3}t^3 + 3t + 1,\ -7\sin t + 8t + 1,\ \tfrac{3}{2}t^2\right\rangle.$$

§Newtow's Second Law

If F is the **force** acting and m is the mass of the particle, then

$$\text{(the force) } F = ma,$$

where a is the acceleration. $F(c(t)) = m\,c''(t)$.



$F = ma.$

Example 2. *Suppose a particle of mass m moves along the path*

$$r(t) = \langle 71 + 5t^2,\ \cos(e^{2t}),\ \ln(t+1)\rangle.$$

Find the force acts on this particle at time $t > 0$.

$$F = m\, r''(t)$$

$$= m \left\langle 10,\ -\cos(e^{2t})\, 4(e^{2t})^2 - 4\sin(e^{2t})\, e^{2t},\ \frac{-1}{(t+1)^2} \right\rangle.$$

§Arc Length

If $c(t) = \langle x(t), y(y), z(t)\rangle$ is a parametrization of a curve in $\mathbb{R}^3$, then the **length** of the curve from t_0 to t_1 is

$$\begin{aligned} L &= \int_{t_0}^{t_1} \|c'(t)\| dt \\ &= \int_{t_0}^{t_1} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt \end{aligned}$$

where L is also called **arc length**.

Example 3. *Find (arc) length of the curve that is parametrized by*

$$c(t) = \langle \cos(t), \sin(t), 2t^{3/2}\rangle,$$

$0 \le t \le 11$.

$$\int_0^{11} \| c'(t) \| \, dt. \quad , \quad c'(t) = (-\sin t, \cos t, 3t^{\frac{1}{2}})$$

$$= \int_0^{11} \sqrt{\underbrace{\cos^2 t + \sin^2 t}_{1} + 9t}\, dt.$$

$$= \int_0^{11} \sqrt{1 + 9t}\, dt. \quad u = 1 + 9t.$$

$$= \frac{1}{9}\frac{2}{3}(1 + 9t)^{\frac{3}{2}} \Big|_0^{11}.$$

$$= \underline{74} \ \#$$