Math 2374 Spring 2018- Week 7

Quick Review from last week

Section 5.5:

(1) Triple integral

$$\int\int\int_W f(x,y,z)dV.$$

In particular, if $f=1$, then $\int\int\int_W f(x,y,z)dV$ is the volume of the region W.

(2) **Shadow method:**
Imagine a sun is on z axes.

$$\int\int\int_W f(x,y,z)dV = \int\int_{shadow}\left(\int_{bottom(x,y)}^{top(x,y)} f(x,y,z)dz\right)dxdy.$$

Section 4.1-4.2:

(1) Suppose $c(t)$ is the path of an object. Then $v(t)=c'(t)$ is its velocity and $a(t)=c''(t)$ is the acceleration.

(2) $F=ma$.

(3) Arc length

$$\begin{aligned} L &= \int_a^b \|c'(t)\|dt \\ &= \int_a^b \sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}dt \end{aligned}$$

* Quiz 5 : 4.1 - 4.4

4.3, 4.4 Vector fields, Divergence, and Curl

A **vector field** in $\mathbb{R}^2$ is a function $F: \mathbb{R}^2 \to \mathbb{R}^2$ that assigns to each point in $\mathbb{R}^2$ a vector in $\mathbb{R}^2$.

F

point

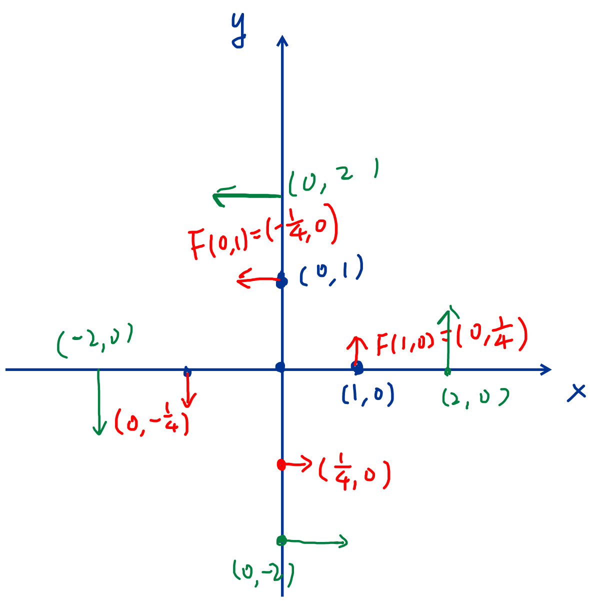
We can write the component functions of F as follows:

(x, y) $\quad (F_1(x,y), F_2(x,y))$

$$F(x, y) = \langle F_1(x, y),\ F_2(x, y)\rangle.$$

*Note that vector fields in three dimensions ($\mathbb{R}^3$), $F: \mathbb{R}^3 \to \mathbb{R}^3$, can be similarly defined.

Example 1. $F(\overbrace{x, y}^{\text{position}}) = \overbrace{\langle -\frac{1}{4}y,\ \frac{1}{4}x\rangle}^{\text{vector}}$. *See also page 237 in textbook.*



$F(0,0) = (0,0)$

$F(1,0) = (0, \frac{1}{4})$

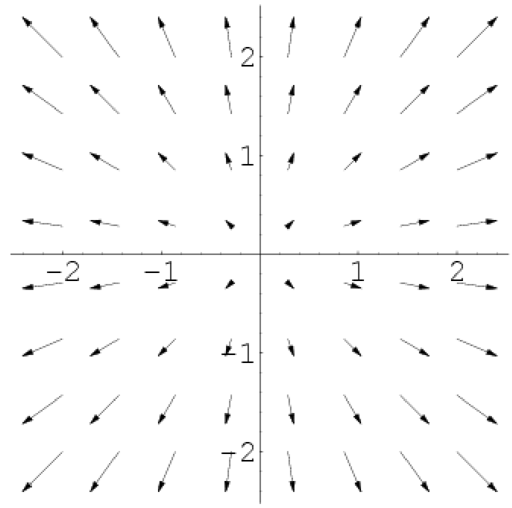
$F(0,1) = (-\frac{1}{4}, 0)$

$F(-1,0) = (0, -\frac{1}{4})$

$F(0,-1) = (\frac{1}{4}, 0)$.

One can think such a vector field represents fluid flow in 2 dimensions. Thus $F(x, y)$ gives the **velocity** of a fluid at the position (x, y). We call $F(x, y)$ the **velocity field of the fluid**.

Example 2. $F(x,y) = \langle \frac{1}{4}x,\ \frac{1}{4}y \rangle$.



§Gradients vector fields

Let $f:\ \mathbb{R}^3 \to \mathbb{R}^1$, the gradient of f is

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z} \right\rangle$$

This is an example of vector field, it assigns a vector to each point (x,y,z).

§Divergence and Curl

For a function $f:\mathbb{R}^1 \to \mathbb{R}^1$, we can think $\frac{d}{dx}$ as an operator:

$$\frac{d}{dx}\underbrace{f}_{input} = \underbrace{f'(x)}_{output}$$

Similarly, we can think ∇ as an operator

$$\nabla \underbrace{f}_{input} = \underbrace{\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}}_{output} = \left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z} \right\rangle$$

∇: called "nabla" operator

"Del"

Recall: $i = \langle 1, 0, 0 \rangle$

$j = \langle 0, 1, 0 \rangle$

$k = \langle 0, 0, 1 \rangle$,

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$.

Definitions:

1. **Divergence** of a vector field F is the dot product of ∇ and F.
   More precisely, if $F = \langle F_1, F_2, F_3 \rangle$, the **divergence** of F is the scalar field

$$\nabla \cdot F = \operatorname{div}(F) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_1, F_2, F_3) = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3.$$

   More generally, if $F = \langle F_1, F_2, \cdots, F_n \rangle$ is a vector field of $\mathbb{R}^n$, the **divergence** of F is

$$\nabla \cdot F = \operatorname{div}(F) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right) \cdot (F_1, F_2, \dots, F_n)$$
$$= \frac{\partial}{\partial x_1}F_1 + \frac{\partial}{\partial x_2}F_2 + \cdots + \frac{\partial}{\partial x_n}F_n.$$

- $\nabla \cdot F = \operatorname{div}(F)$ is scalar-valued.
- $\nabla \times F = \operatorname{curl}(F)$ is a vector-valued

2. **Curl** of a vector field F is the cross product of ∇ and $F = \langle F_1, F_2, F_3 \rangle$.

$$\nabla \times F = \operatorname{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$
$$= \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle.$$

Example 3. *Let* $F = \langle x^2 + e^x y,\ xy,\ x + z\rangle$. *Find* $\mathrm{div}F$ *and* $\mathrm{curl}F$.

$$\mathrm{div}\, F = \nabla \cdot F.$$

$$= \frac{\partial}{\partial x}(x^2 + e^x y) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(x+z)$$

$$= 2x + e^x y + x + 1 = \underline{e^x y + 3x + 1}. \ \#$$

$$\mathrm{curl}\, F = \nabla \times F$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + e^x y & xy & x + z \end{vmatrix}$$

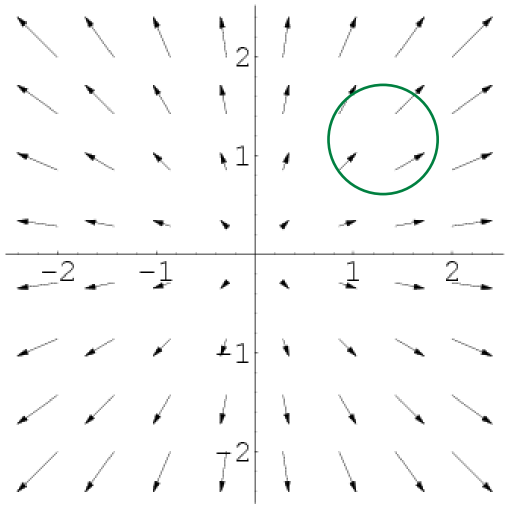
$$= \langle 0 - 0,\ 0 - 1,\ y - e^x \rangle. \ \#$$

§Physical Interpretations

Imagine F is the velocity field of a fluid (or a gas). This can help understanding properties of basic vector fields, such as divergence, $\nabla\cdot$, and curl, $\nabla\times$.

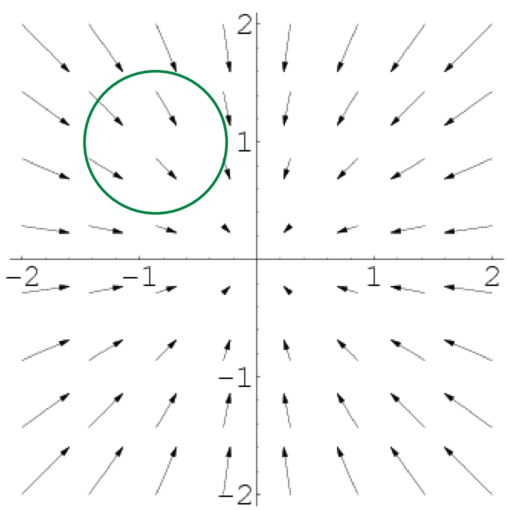
Example 4. *Consider the vector field* $F = \langle \frac{x}{4}, \frac{y}{4} \rangle$.

$$\operatorname{div} F = \frac{\partial}{\partial x}\left(\frac{x}{4}\right) + \frac{\partial}{\partial y}\left(\frac{y}{4}\right) = 1/2 . > 0.$$



Example 5. *Consider the vector field* $F = \langle -\frac{x}{4}, -\frac{y}{4} \rangle$.

$$\operatorname{div} F = -1/2 < 0.$$



Conclusions:

(1) div$F$: the rate of expansion per unit volume

- div$F < 0$, the fluid is compressing.
- div$F > 0$, the fluid is expanding.

§Physical Interpretations

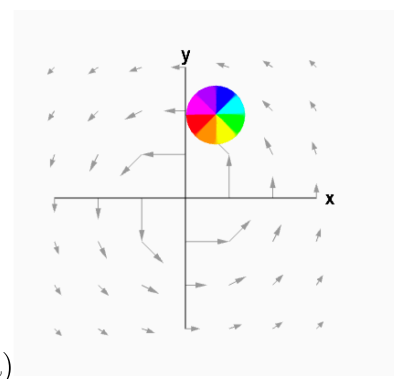
Imagine F is the velocity field of a fluid (or a gas). The curl F captures the idea of how a fluid (or a gas) may rotate.

(2) curl$F$:

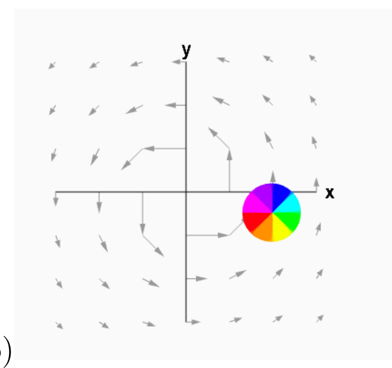
- curl$F = 0$, the paddle wheel does NOT spin.

If curl$F = 0$, we call the vector field F is **irrotational**.

Example 6. *See examples in math insight entitled "Subtleties about curl". Show* $V(x,y,z) = \frac{1}{x^2+y^2}(-y\boldsymbol{i} + x\boldsymbol{j})$ *is irrotational when* $(x,y) \neq (0,0)$.



(a)



(b)

curl V $(\nabla \times V)$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = \left\langle 0-0,\ \frac{\partial}{\partial z}\left(\frac{-y}{x^2+y^2}\right) - 0,\ \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \right\rangle$$

$$= \left\langle 0,\ 0,\ \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+y(2y)}{x^2+y^2} \right\rangle$$

$$= \langle 0, 0, 0 \rangle.$$

Fact. *We have the following two facts:*

- *Gradients are curl free:*

$$\boxed{\nabla \times (\nabla f) = 0.} \quad (f \textit{ is a scalar valued function})$$

$$\nabla \times \nabla f = \nabla \times (f_x, f_y, f_z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \langle 0, 0, 0 \rangle.$$

- *Curls are divergence free:*

$$\boxed{\operatorname{div}(\operatorname{curl} F) = \nabla \cdot (\nabla \times F) = 0.} \quad (F \textit{ is a vector valued function})$$

$F = \langle F_1, F_2, F_3 \rangle$

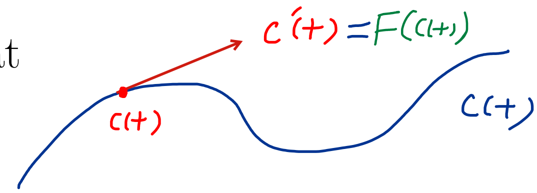
$$\nabla \times F = \left\langle \frac{\partial}{\partial y} F_3 - \frac{\partial}{\partial z} F_2,\ \frac{\partial}{\partial z} F_1 - \frac{\partial}{\partial x} F_3,\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle.$$

$\Rightarrow \nabla \cdot (\nabla \times F) = 0.$

§Flow lines

A **flow line** for a vector field F is a path $c(t)$ such that

$$c'(t) = F(\underbrace{c(t)}_{\text{position}}).$$



In other words, the tangent vector $c'(t)$ of the curve coincides with the vector field $F(c(t))$.

Example 7. *Show that* $c(t) = (\overbrace{r\sin(t)}^{=x}, \overbrace{r\cos(t)}^{=y}, \overbrace{e^t}^{=z})$ *is a flow line for the vector field* $F(x, y, z) = (y, -x, z)$.

$c'(t) = \langle r\cos t, -r\sin t, e^t \rangle.$

$F(c(t)) = \langle r\cos t, -r\sin t, e^t \rangle.$

Thus, $F(c(t)) = c'(t)$, it implies $c(t)$ is a flow line for F.

7.1 Line integral of a scalar-valued function

In this section, we consider a scalar function $f : \mathbb{R}^3 \to \mathbb{R}^1$ and a path $c(t) : [a, b] \to \mathbb{R}^3$.

Definition:

The **line integral** of a scalar-valued function f along the path $c(t),\ a \leq t \leq b$, is defined to be

$$\int_c f ds = \int_a^b f(c(t))\|c'(t)\|dt.$$

Remark:

1. If we let $f(x, y, z)$ denote the mass density at (x, y, z) and suppose the image of $c(t)$ represents a wire, then

$$\int_c f ds = \int_a^b f(c(t))\|c'(t)\|dt$$

can be viewed as the **total mass** of the wire.

2. If $f = 1$, then

$$\int_c f ds = \int_a^b \|c'(t)\|dt$$

is the arc length.

$$\int 1\, ds = \int_a^b \|c'(t)\|\, dt.$$

Example 1. *Let* $f(x,y,z) = x^2+y^2+z^2$ *and* $c(t) = (\overbrace{\cos(t)}^{=x}, \overbrace{\sin(t)}^{=y}, \overbrace{t}^{=z}), 0 \le t \le 2\pi$. *Find* $\int_c f\,ds$.

$$\int_C f\,ds = \int_0^{2\pi} f(c(t))\,\|c'(t)\|\,dt$$

$$= \int_0^{2\pi} \big(\underbrace{(\cos^2 t) + \sin^2 t}_{=1} + t^2\big)\,\|\langle -\sin t, \cos t, 1\rangle\|\,dt$$

$$= \int_0^{2\pi} (1+t^2)\sqrt{2}\,dt$$

$$= 2\sqrt{2}\,\pi + \frac{8\sqrt{2}}{3}\pi^3 .$$

#

Example 2. *If the path* $c(t) = (\overbrace{\sin^2(t)}^{x}, \overbrace{\cos^2(t)}^{=y}), 0 \le t \le \pi/2$ *represents a wire with density at the point* (x,y) *given by* $f(x,y) = y$ *grams per unit length. Find the total mass of the wire.*

$$\int_C f\,ds = \int f(c(t))\,\|c'(t)\|\,dt$$

$$= \int_0^{\pi/2} \cos^2 t\,\|\langle 2\cos t\sin t, 2\cos(t)(-\sin t)\rangle\|\,dt$$

$$= \int_0^{\pi/2} \cos^2 t\,\sqrt{8\cos^2 t\sin^2 t}\,dt$$

$$= \int_0^{\pi/2} \cos^2 t\;2\sqrt{2}\,\sin t\cos t\,dt$$

$$= 2\sqrt{2}\int_0^{\pi/2} \sin t\cos^3 t\,dt \qquad u = \cos t.$$

$$= 2\sqrt{2}\left(-\frac{1}{4}\cos^4 t\right)\Big|_0^{\pi/2}$$

$$= \frac{\sqrt{2}}{2} .$$

#