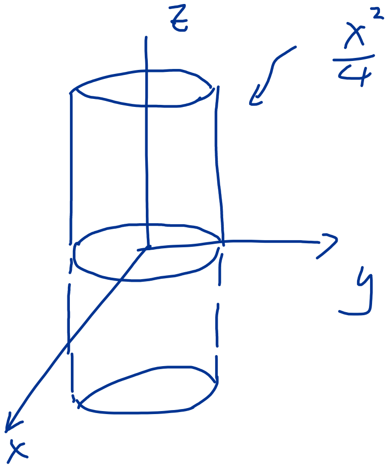
Example 3. *1. Find the intersection of the plane $z = x$ and the cylinder $\frac{x^2}{4} + \frac{y^2}{9} = 1$.*



$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Recall:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1.$$

$$(x, y) = (a\cos\theta, a\sin\theta)$$

$$0 \le \theta \le 2\pi.$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$$

$$x = 2\cos\theta$$

$$y = 3\sin\theta, \quad 0 \le \theta \le 2\pi.$$

$$c(\theta) = (\underbrace{2\cos\theta}_{x}, \underbrace{3\sin\theta}_{y}, \underbrace{\overset{z=x}{2\cos\theta}}_{z}).$$

2. *Suppose the curve that is the intersection of the above two surfaces represents a wire with density at the point (x, y, z) given by $f(x, y, z) = yx + 100$ grams per unit length. Set up the integral that represents the total mass of the wire.*

$$\text{Total mass} = \int_C f\, ds = \int f(c(\theta))\, \|c'(\theta)\|\, d\theta.$$

$$= \int_0^{2\pi} \left(100 + 2\cos\theta\,(3\sin\theta)\right) \|(-2\sin\theta, 3\cos\theta, -2\sin\theta)\|\, d\theta.$$

$$= \int_0^{2\pi} \left(100 + 6\cos\theta\sin\theta\right) \sqrt{8\sin^2\theta + 9\cos^2\theta}\, d\theta.$$

#

§7.2 Line integrals of a vector field

In this section, we consider the problem of integrating a vector field along a path.

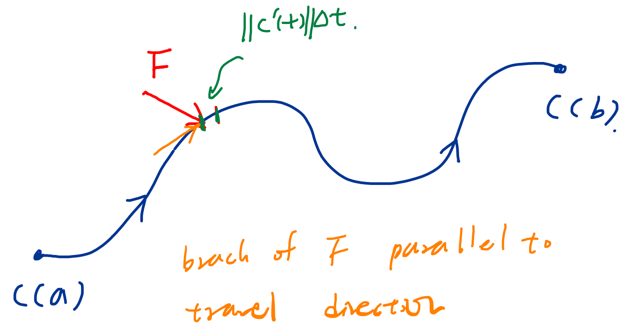
Definition:
The **line integral** of a vector field F along the curve C that is parametrized by $c(t),\ a \le t \le b$, is defined to be

$$\int_C F \cdot d\mathbf{s} = \int_a^b F(c(t)) \cdot c'(t)dt.$$

Motivation: (Work done by force F)

Recall : work = Force · displacement.

Suppose a particle moves along a path $c(t)$, $a \le t \le b$, this particle is experiencing the force $F(x,y,z)$, at position (x, y, z).



* on small piece, the work is $\left(F \cdot \frac{c'(t)}{\|c'(t)\|}\right)\|c'(t)\|\Delta t$

brach of F parallel to travel direction is $F \cdot \frac{c'(t)}{\|c'(t)\|}$

So the total work along C by force F is

$$\int F \cdot c'(t)\, dt = \int_C F \cdot d\mathbf{s}$$

Example 4. *Let* $F = \langle x^2,\ -xy,\ z\rangle$*. Suppose the curve* C *is parametrized by* $c(t) = (\underbrace{\sin(t)}_{x}, \underbrace{\cos(t)}_{y}, \underbrace{t}_{z}), 0 \le t \le \pi$*. Find* $\int_C F \cdot d\boldsymbol{s}$*.*

$$\int_C F \cdot d\vec{s} = \int_0^{\pi} F(c(t)) \cdot c'(t)\, dt$$

$$= \int_0^{\pi} (\sin^2 t,\ -\sin t \cos t,\ t) \cdot (\cos t, -\sin t, 1)\, dt$$

$$= \int_0^{\pi} (\sin^2 t \cos t + \sin^2 t \cos t + t)\, dt.$$

$$= \int_0^{\pi} (2\sin^2 t \cos t + t)\, dt$$

$\downarrow u = \sin t$

$$= 2\tfrac{1}{3}\sin^3 t + \tfrac{1}{2}t^2 \Big|_0^{\pi} = \tfrac{1}{2}\pi^2 \quad \#$$

§Differential form

Let the path $c(t) = (x(t), y(t), z(t))$. We write

$$F(x, y, z) = (P(x, y, z),\ Q(x, y, z),\ R(x, y, z))$$

and also write $d\mathbf{s}$ as the differential form

$$d\mathbf{s} = (dx,\ dy,\ dz),$$

then we can rewrite

$$\int_c F \cdot d\mathbf{s} = \int_c (P, Q, R) \cdot (dx, dy, dz)$$

$$= \int_c P\,dx + Q\,dy + R\,dz. \quad \text{— } ①$$

On the other hand,

$$\int_c F \cdot ds = \int F(c(t)) \cdot c'(t)\,dt.$$

$$= \int (P(c(t)), Q(c(t)), R(c(t))) \cdot (x'(t), y'(t), z'(t))\,dt$$

$$= \int P(c(t))\,x'(t) + Q(c(t))\,y'(t) + R(c(t))\,z'(t)\,dt. \quad ②$$

So ① = ②

Example 5. *Evaluate the line integral*

$$\int_C x^2 dx + xydy + dz,$$

where the curve C is parametrized by $c(t) = (\overbrace{t}^{=x}, \overbrace{t^2}^{=y}, \overbrace{1}^{=z}),\ 0 \le t \le 1.$

[method 1] Consider the vector field

$$F = \langle x^2, xy, 1 \rangle.$$

Then

$$\int_C x^2\,dx + xy\,dy + dz = \int_C F \cdot ds$$

$$= \int F(c(t)) \cdot c'(t)\,dt$$

$$= \int_0^1 (t^2, t^3, 1) \cdot (1, 2t, 0)\,dt$$

$$= \int_0^1 t^2 + 2t^4\,dt = {}^{11}/_{15}. \ \#$$

[method 2]

$$\int_C \overbrace{x^2}^{P}\,dx + \overbrace{xy}^{Q}\,dy + \overbrace{1}^{R}dz \qquad \begin{cases} x(t) = t \\ y(t) = t^2 \\ z(t) = 1. \end{cases}$$

$$= \int \left(t^2\, x'(t) + t^3\, y'(t) + 1\ z'(t)\right) dt$$

$$= \int \left(t^2\ 1 + t^3 (2t) + 1 \cdot 0\right) dt$$

$$= \int_0^1 t^2 + 2t^4\,dt = {}^{11}/_{15}. \ \#$$

§Reparametrization

Suppose that a curve C is parametrized by

$$c(t), \ a \leq t \leq b$$

and also parametrized backward by

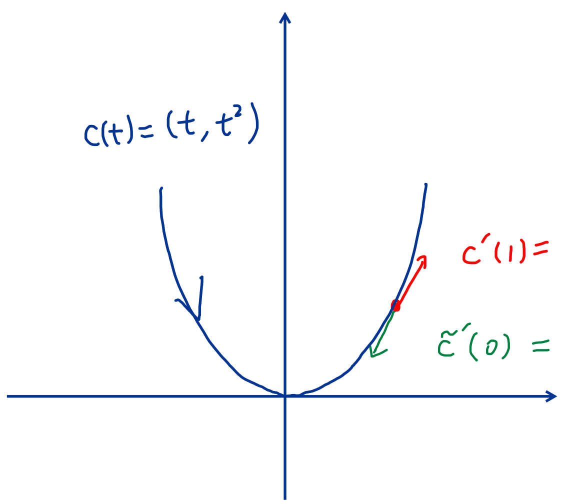
$$c_{-}(t) = c(a + b - t).$$

A simple curve C has two orientations (we discussed in section 2.4) that are determined by unit tangent vectors

$$T = \frac{c'(t)}{\|c'(t)\|} \quad \text{and} \quad T^{-} = \frac{c'_{-}(t)}{\|c'_{-}(t)\|} \text{ (that points in opposite direction.)}$$

$c(t) = (t, t^2)$, $-1 \leq t \leq 2$

$\tilde{c}(t) = (1-t, (1-t)^2)$, $-1 \leq t \leq 2$.



$c'(1) = \dfrac{\langle 1, 2 \rangle}{\sqrt{5}}$

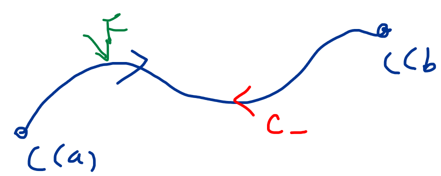
$\tilde{c}'(0) = \dfrac{\langle -1, -2 \rangle}{\sqrt{5}}$

NOTE: $c(1) = \tilde{c}(0)$ the same point.

Thus,

1. Line integrals $\int_c F \cdot d\mathbf{s}$ (can be interpreted as Work done by force F):

$$\int_{c_-} F \cdot d\mathbf{s} = -\int_c F \cdot d\mathbf{s}$$



2. Path integrals $\int_c f ds = \int f(c(t))\|c'(t)\|dt$ (can be interpreted as Total mass of wire):

$$\int_{c_-} f ds = \int_c f ds$$

Example 6. *Consider a curve parametrized by*

$$c(t) = (t, t^2),\ 0 \le t \le 1.$$

Then the same curve with the opposite orientation is as follows:

$$\tilde{c}(\tilde{t}) = (1 - \tilde{t}, (1 - \tilde{t})^2),\ 0 \le \tilde{t} \le 1.$$

Let $f(t) = 8t$ and vector field $F(x, y) = (x + y,\ x)$. Then

(1)

$$\int_c F \cdot d\mathbf{s} = \int_0^1 (t + t^2, t) \cdot (1, 2t) dt = 3/2$$

and

$$\int_{\tilde{c}} F \cdot d\mathbf{s} = \int_0^1 ((1 - \tilde{t}) + (1 - \tilde{t})^2, 1 - \tilde{t}) \cdot (-1, 2(1 - \tilde{t})) d\tilde{t} = -3/2.$$

(2)

$$\int_c f ds = \frac{2}{3}(5^{3/2} - 1)$$

and

$$\int_{\tilde{c}} f ds = \frac{2}{3}(5^{3/2} - 1).$$