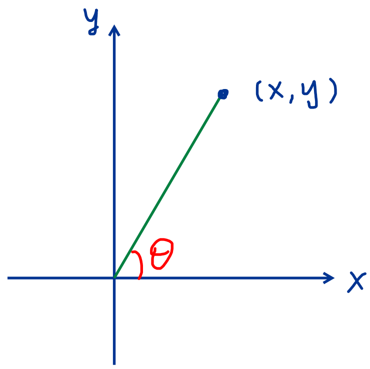
Review:

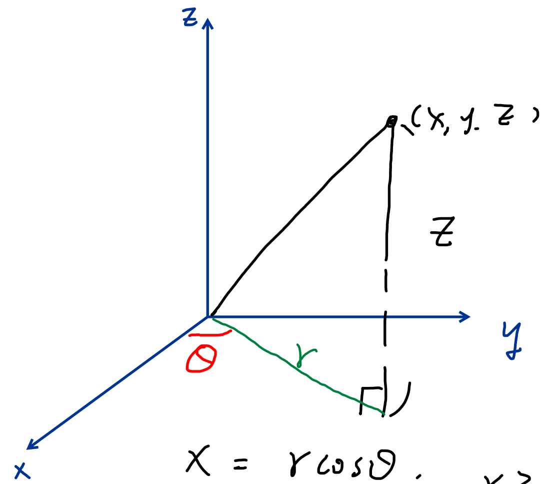
§ Polar coordinate:



$x = r\cos\theta$, $0 \le r$

$y = r\sin\theta$, $0 \le \theta < 2\pi$

§ Cylindrical coordinate:

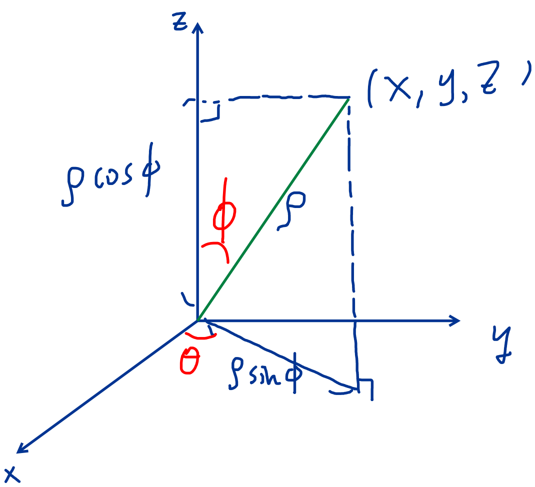


$x = r\cos\theta$, $\quad r \ge 0$

$y = r\sin\theta$, $\quad 0 \le \theta < 2\pi$

$z = z$,

§ Spherical coordinate:



$x = (\rho\sin\phi)\cos\theta$, $\quad \rho \ge 0$

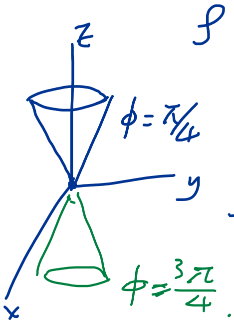
$y = (\rho\sin\phi)\sin\theta$, $\quad 0 \le \theta < 2\pi$

$z = \rho\cos\phi$, $\quad 0 \le \phi \le \pi$

Example: Express $x^2+y^2-z^2=0$ in spherical coordinates.

$$\underline{\rho^2\sin^2\phi}\cos^2\theta + \underline{\rho^2\sin^2\phi}\sin^2\theta - \rho^2\cos^2\phi = 0.$$

$$\rho^2\sin^2\phi\,\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} - \rho^2\cos^2\phi = 0$$



$$\rho^2(\sin^2\phi - \cos^2\phi) = 0$$

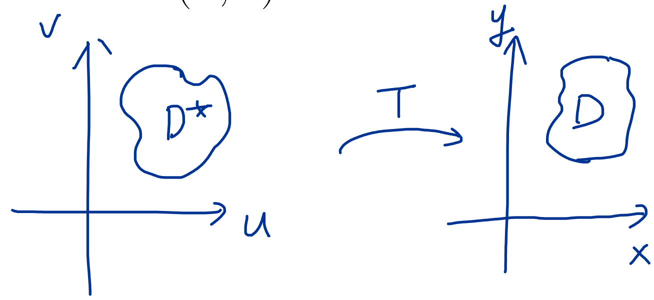
$\rho = 0$ or $\sin^2\phi = \cos^2\phi \Rightarrow \sin\phi = \pm\cos\phi$, $0 \le \phi \le \pi$,

$\Rightarrow \phi = \pi/4 \; , \; 3\pi/4$

6.1- 6.2 The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ and the change of variables theorem

We consider a function T that maps some region D^* in the (u, v) coordinates into the origin region D in (x, y) coordinates, that is,

$$T : D^* \to D$$



Then we denote

$$(x, y) = T(u, v).$$

Example 4. *Let* $D^* = [0, 1] \times [0, 2\pi]$, *a rectangle in* $\mathbb{R}^2$. *Let* $T(r, \theta) = (r\cos\theta,\ r\sin\theta)$. *What is the image set* $D = T(D^*)$?[1]

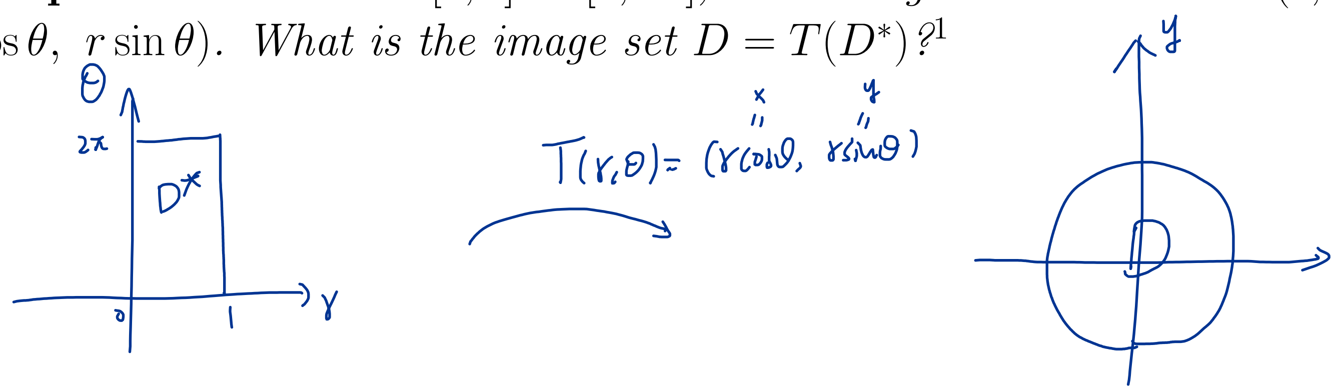


Image of $T = D$ is a disk with radius 1 centered at $(0,0)$.

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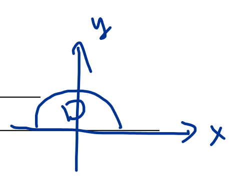
[1] If $D^* = [0, 1] \times [0, \pi]$, then D is ________



§Images of Maps T.

Let T be the linear mapping of $\mathbb{R}^2$ to $\mathbb{R}^2$ given by

$$T(\vec{x}) = A\vec{x},$$

where a point $\vec{x}$ in $\mathbb{R}^2$ expressed by

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix},$$

and a 2×2 matrix with $\det(A) \neq 0$ denoted by

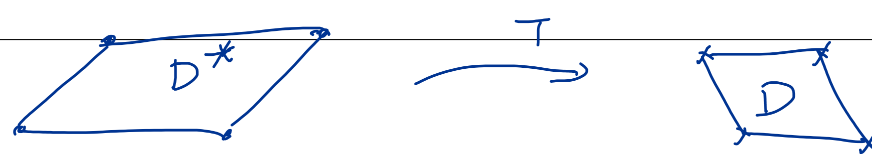
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then we can further express det $A \neq 0$.

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

Fact. *T maps **parallelograms** into **parallelograms** and **vertices** into **vertices**.*

On the other hand, if the image $T(D^*)$ is a parallelogram, then D^* must be a parallelogram.



Example 5. *Let*

$$T(u,v) = \left(\frac{u+v}{2},\ \frac{u-v}{2}\right)$$

and $D^* = [-1,1]\times[-1,1]$ *in* $\mathbb{R}^2$. *Find the image set* $D = T(D^*)$.

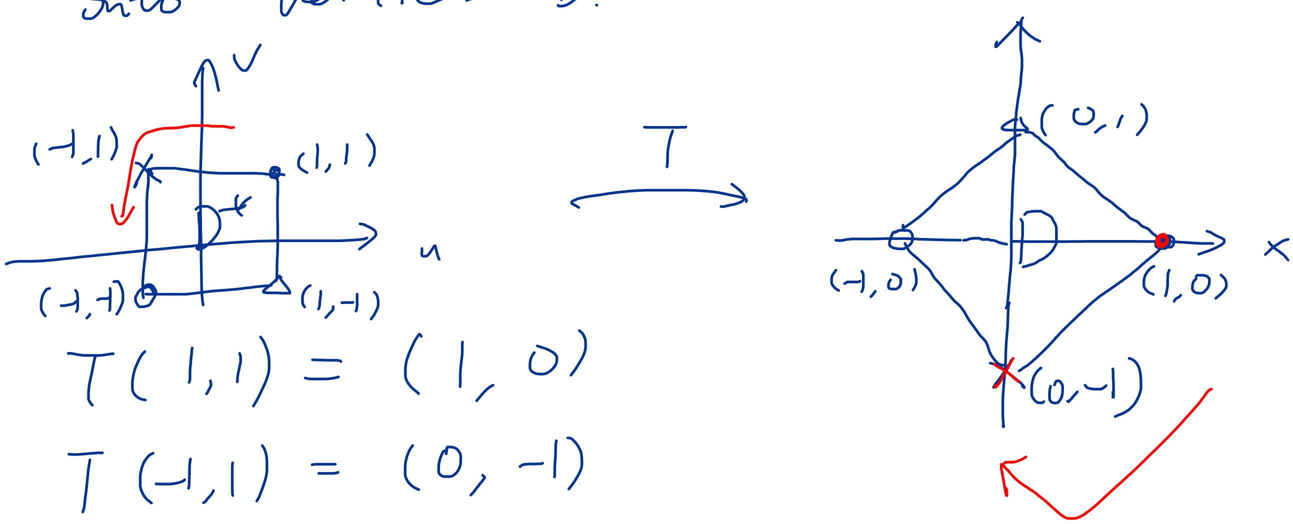
$(a,b) \cdot (u,v)$

$$T(u,v) = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}_{2\times 2}}_{A} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \boxed{\frac{u+v}{2}} \\ \frac{u-v}{2} \end{bmatrix}.$$

[See NOTE 0/26. det $A < 0$. Then T revese the orientation]

Check $\det A = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2} \neq 0$.

By Fact, T maps parallelograme D^* onto parallelogram D. and vertices of D^* onto vertices D.



$T(1,1) = (1,0)$

$T(-1,1) = (0,-1)$

$T(-1,-1) = (-1,0)$

$T(1,-1) = (0,1)$

Example 6. *Suppose the parallelogram D is bounded by the lines*

$$y = \frac{3}{2}x - 4,\ \ y = \frac{3}{2}x + 2,\ \ y = -2x + 1,\ \ y = -2x + 3.$$

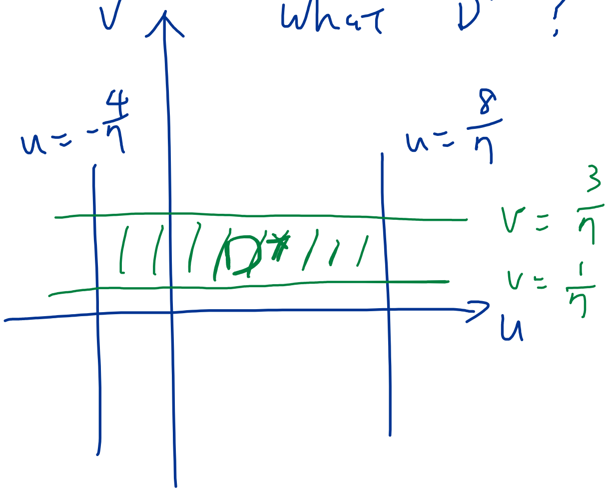
Consider a map T that maps D^* into D and is defined by

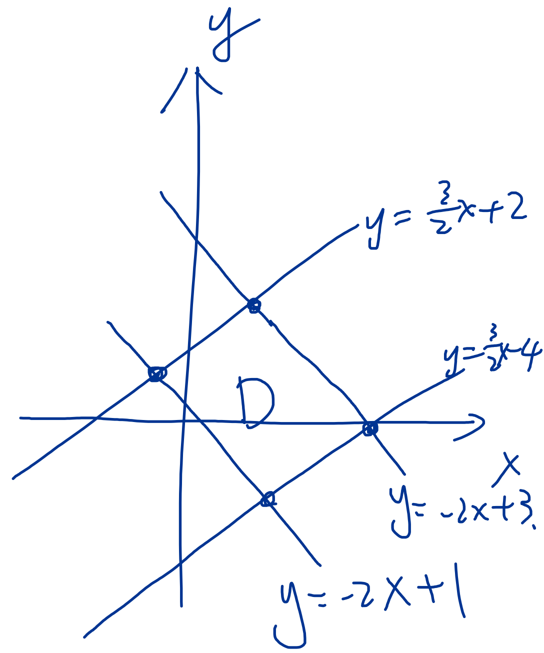
$$(x, y) = T(u, v) = (u + 2v,\ \ -2u + 3v).$$

$$= \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

What is the region D^*?

What D^* ?





Find equations in terms of u, v.

$$\begin{cases} 2y = 3x - 8 \\ 2y = 3x + 4 \\ y = -2x + 1 \\ y = -2x + 3 \end{cases} \Longrightarrow \begin{cases} 2(-2u+\cancel{3v}) = 3(u+\cancel{2v}) - 8 \\ 2(-2u+\cancel{3v}) = 3(u+\cancel{2v}) + 4 \\ -\cancel{2u}+3v = -2(\cancel{u}+2v) + 1 \\ -\cancel{2u}+3v = -2(\cancel{u}+2v) + 3 \end{cases}$$

$$\Rightarrow \begin{cases} 7u = 8 \\ 7u = -4 \\ 7v = 1 \\ 7v = 3 \end{cases}$$

$$D^*: \ -\frac{4}{7} \le u \le \frac{8}{7}$$
$$\frac{1}{7} \le v \le 3/7. \quad \#$$

The Change of Variables Theorem

Recall: In Cal. 1, we did the follows computations:

$$\int_0^{\sqrt{\pi}} 2x\sin(x^2)dx,$$

$u = x^2$

$du = 2x\,dx$

$= \int 2x\sin(x^2)\left|\frac{dx}{du}\right| du$

$= \int 2x\sin(u)\underbrace{\left|\frac{1}{2x}\right|}_{\text{Jacobian}} du$

we apply u-sub,

$$u = x^2, \quad du = 2xdx$$

so we have

$$\int_0^{\pi} \sin(u)du,$$

where $u = x^2$ maps $[0, \sqrt{\pi}]$ into $[0, \pi]$.

One motivation to study "Change of variables", is to transform the region of integration so that the resulting integral becomes easier to solve.

§Change of variables for double integrals

Fact. *Let D^* and D be elementary regions in $\mathbb{R}^2$. Let T maps D^* onto D is given by*

$$\underline{T(u,v) = (x(u,v), y(u,v)).}$$

Then

$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(x(u,v), y(u,v)) \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{|\det DT|} dudv.$$

Here the determinant of the derivative matrix

$$det\underbrace{\boldsymbol{D}}_{\text{matrix of partial derivatives.}}T = \frac{\partial(x,y)}{\partial(u,v)} = det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix},$$

the ***Jacobian*** *of T.*

Example 7. *Consider the map T which transforms polar coordinates into Cartesian coordinates. Then $T(r,\theta) = (r\cos\theta,\ r\sin\theta)$, that is,*

$$x = r\cos\theta, \quad y = r\sin\theta.$$

What is Jacobian of T?