Math 2374 Spring 2018 - Week 12

Quick Review from previous lecture

1. Tangent vectors: $\overset{\overset{T_u}{\shortparallel}}{\frac{\partial \mathbf{\Phi}}{\partial u}}$ and $\overset{\overset{T_v}{\shortparallel}}{\frac{\partial \mathbf{\Phi}}{\partial v}}$.

2. The unit normal vector is $\mathbf{n} = \frac{\frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v}}{\| \frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v} \|}$. Positive side of surface is the side with normal $\mathbf{n}$.

3. If $\frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v} \neq 0$, then the tangent plane of the surface at $\mathbf{\Phi}(u_0, v_0) = (x_0, y_0, z_0)$ is
$$\boxed{\left(\frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v}\right)(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.}$$

4. $\text{Area}(S) = \iint_D || \frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v} || \, dudv.$

5. Let $\Phi(u, v) = (x(u, v),\ y(u, v),\ z(u, v))$ be a parametrization of the surface S.

   Surface integral of a scalar-valued function:
$$\boxed{\int \int_S f(x, y, z) dS = \int \int_D f(\Phi(u, v)) \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| dudv.}$$

Example 1. *Evaluate* $\int\int_S (1+4z)dS$, *where* S *is the surface* $x^2+y^2=z$, $0 \le z \le 1$.

Parametrization : $\Phi(x,y) = (x, y, x^2+y^2)$, $0 \le x^2+y^2 \le 1$

$T_x = \langle 1, 0, 2x \rangle$,

$T_y = \langle 0, 1, 2y \rangle$

$T_x \times T_y = \langle -2x, -2y, 1 \rangle$

$\| T_x \times T_y \| = \sqrt{1 + 4x^2 + 4y^2}$

Then $\iint (1+4z) \, \| T_x \times T_y \| \, dx\,dy$

$$= \iint_{x^2+y^2 \le 1} \left(1+4(x^2+y^2)\right) \sqrt{1+4x^2+4y^2} \, dx\,dy$$

Polar coord

$$= \int_0^{2\pi}\int_0^1 (1+4r^2)\sqrt{1+4r^2} \, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^5 \underbrace{u \, u^{1/2}}_{u^{3/2}} \frac{du}{8}\,d\theta$$

$u = 1+4r^2$
$du = 8r\,dr$

$$= \int_0^{2\pi} \frac{\cancel{2}^{\,1}}{5} u^{\frac{5}{2}} \frac{1}{\cancel{8}_{\,4}} \Big|_1^5 \, d\theta$$

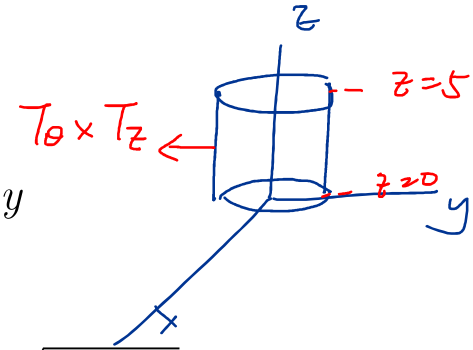
$$= 2\pi \frac{1}{20} \left(5^{\frac{5}{2}} - 1\right)$$

$$= \frac{\pi}{10}\left(5^{\frac{5}{2}} - 1\right). \#$$

§The **total mass** of the surface

Let $m(x,y,z)$ be the mass density function of the surface. Then the **total mass** of surface S is

$$\int\int_S \underbrace{m(x,y,z)}_{density}\,dS.$$



Example 2. *For the cylindrical surface S of given by*

$$x^2+y^2=9, \quad 0 \le z \le 5.$$

Suppose S has a mass density function $m(x,y,z) = z\sqrt{x^2+y^2}$. Find the total mass of the surface.

Total mass $= \iint_S z\sqrt{x^2+y^2}\, dS$

Parametrization : $\Phi(\theta, z) = (3\cos\theta, 3\sin\theta, z)$

$0 \le \theta < 2\pi, \quad 0 \le z \le 5.$

$T_\theta = (-3\sin\theta, 3\cos\theta, 0)$

$T_z = (0, 0, 1)$

$T_\theta \times T_z = (3\cos\theta, 3\sin\theta, 0)$

$\|T_\theta \times T_z\| = 3.$

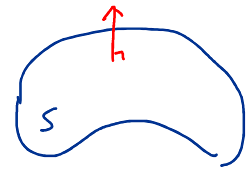
Total mass $= \int_0^5\int_0^{2\pi} z \overbrace{\sqrt{3^2\cos^2\theta + 3^2\sin^2\theta}}^{3} \overbrace{\|T_\theta \times T_z\|}^{3} d\theta dz$

$= \int_0^5\int_0^{2\pi} z \cdot 9 \; d\theta\, dz = 225\pi$ #

7.6 Surface Integrals of Vector Fields

Recall:

- The **line** integral of a vector field F:
  $\int_\gamma F \cdot d\mathbf{s}$ could be interpreted as the work done by the force field F on a particle moving along the curve γ.



[Motivation: Surface Integrals of Vector Fields]

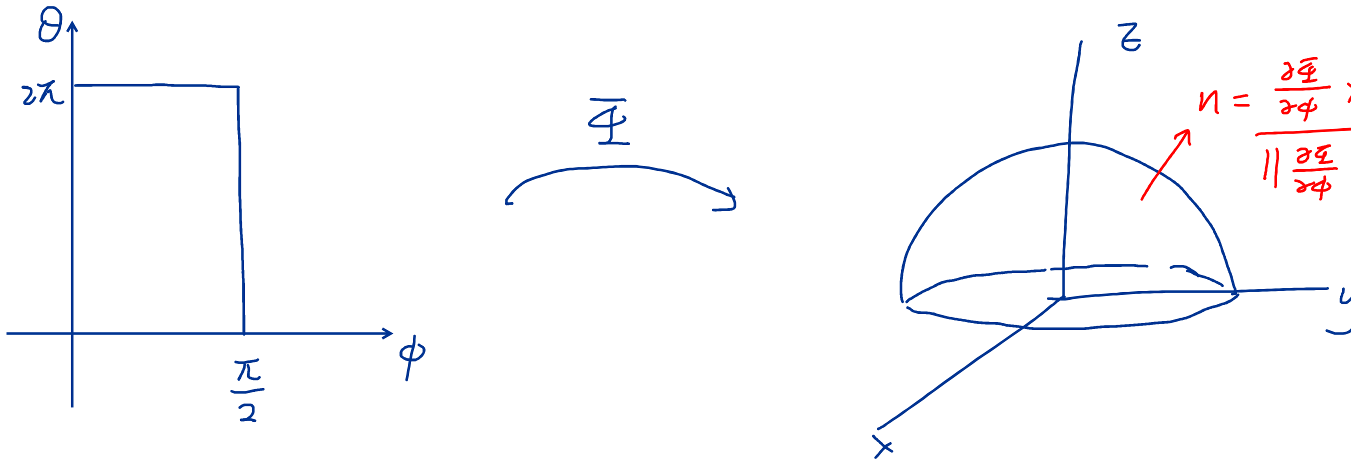
- View F as a velocity vector field of a fluid (flow of a fluid).
  The **surface** integral of a vector field F is
  the amount of fluid flowing through the surface S (per unit time),

  or the flux of fluid through the surface.

Simple observations:

- If water is flowing perpendicular to the surface, then a lot of water will flow through the surface and the flux will be large.
- If water is flowing parallel to the surface, then the flux pass through S is $\underline{0}$.

$\underline{EX:}$ $\Phi(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$



$$n = \frac{\frac{\partial \Phi}{\partial \phi} \times \frac{\partial \Phi}{\partial \theta}}{\left\| \frac{\partial \Phi}{\partial \phi} \times \frac{\partial \Phi}{\partial \theta} \right\|}$$

$\Rightarrow$ Total amount of water flowing through the surface S is to add up the "$F \cdot n$"

scalar

[Surface integrals of a vector field:]

$$\iint_S \overbrace{(F\cdot n)}\, dS = \iint (F\cdot n)\,\|T_u\times T_v\|\,du\,dv.$$

$$= \iint F\cdot\left(\frac{T_u\times T_v}{\cancel{\|T_u\times T_v\|}}\right)\cancel{\|T_u\times T_v\|}\,du\,dv$$

$$= \iint_S F\cdot(T_u\times T_v)\,du\,dv$$

$$\text{Flux} = \iint_S F\cdot dS = \iint_S F\cdot(T_u\times T_v)\,du\,dv.$$

#

Let D be an elementary region. Let $\Phi:\ D\ \to\ \mathbb{R}^3$ be written as

$$\Phi(u,v) = (x(u,v),\ y(u,v),\ z(u,v)),$$

which is a parametrization of the surface S.

Definition:

The surface integral of a vector field $F(x,y,z)$ over a surface S is defined as

$$\int\int_S F\cdot d\mathbf{S} = \int\int_D F(\Phi(u,v))\cdot\left(\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}\right)dudv. \tag{1}$$

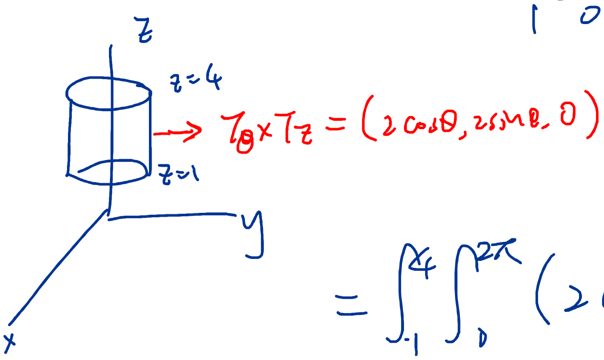
Example 1. *Evaluate* $\int\int_S F \cdot d\boldsymbol{S}$, *where* $F = (x, y, -y^3 + e^y)$ *and* S *is the cylindrical surface defined by* $x^2 + y^2 = 4,\ 1 \le z \le 4$ *with normal points out of the cylinder.*

Parametrization: $\Phi(\theta, z) = (2\cos\theta, 2\sin\theta, z)$,

$0 \le \theta < 2\pi, \quad 1 \le z \le 4.$

$$T_\theta = (-2\sin\theta, 2\cos\theta, 0)$$
$$T_z = (0, 0, 1)$$
$$T_\theta \times T_z = (2\cos\theta, 2\sin\theta, 0) \to \text{normal vector}$$

$$\iint F \cdot dS = \int_1^4 \int_0^{2\pi} F(\Phi(\theta, z)) \cdot (T_\theta \times T_z)\, d\theta\, dz.$$



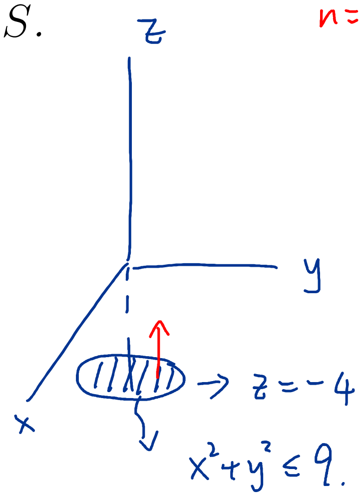
$$= \int_1^4 \int_0^{2\pi} (2\cos\theta, 2\sin\theta, -8\sin^3\theta + e^{2\sin\theta}) \cdot (2\cos\theta, 2\sin\theta, 0)\, d\theta\, dz$$

$$= \int_1^4 \int_0^{2\pi} 4 \, d\theta\, dz = 24\pi$$

Example 2. *Suppose a fluid in* $\mathbb{R}^3$ *with flow given by the vector field* $F(x, y, z) = (0, 0, x^2 + y^2)$. *Suppose that surface* S *is described by* $x^2 + y^2 \leq 9$, $z = -4$, *oriented with an upward pointing normal. Find the total flux through the surface* S.

$n = (a, b, c)$, $c \geq 0$.

$\iint F \cdot dS$



$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, -4)$.

$0 \leq r \leq 3$

$0 \leq \theta < 2\pi$

$T_r = (\cos\theta, \sin\theta, 0)$

$T_\theta = (-r\sin\theta, r\cos\theta, 0)$

$T_r \times T_\theta = (0, 0, r)$, $r \geq 0$

normal is pointing upward

$$\text{Flux} = \iint F \cdot dS = \int_0^{2\pi}\int_0^3 (0, 0, r^2) \cdot (0, 0, r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 r^3 \, dr\, d\theta = 81\pi/2. \#$$