Quick Review from previous lecture

Let $\Phi : D \to \mathbb{R}^3$ be a parametrization of surface S.

- The integral of a real-valued function $f(x, y, z)$ over a surface S is defined as

$$\int\int_S f(x,y,z)dS = \int\int_D f(\Phi(u,v))\|\frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v}\|dudv. \tag{1}$$

  In particular, let $f(x, y, z)$ be the mass density function of the surface. Then the total mass of surface S is

$$\int\int_S f(x,y,z)dS.$$

- The flux of fluid through the surface S is

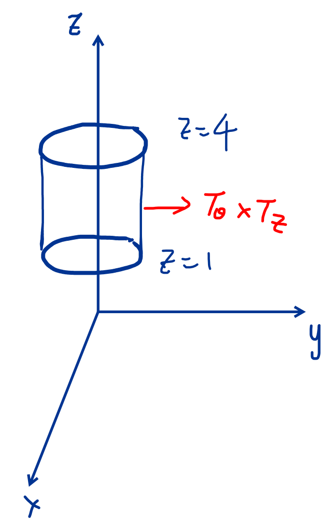
$$Flux = \int\int_S F \cdot d\mathbf{S} = \int\int_D F(\Phi(u,v)) \cdot \left(\frac{\partial \mathbf{\Phi}}{\partial u} \times \frac{\partial \mathbf{\Phi}}{\partial v}\right) dudv. \tag{2}$$

§Independence of Parametrization

Let $\Phi(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$, be a parametrization of the oriented surface S.

We said the parametrization Φ is orientation-preserving(orientation-reversing) parametrization if the vectors $\left(\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v}\right)$ points outside (inside) of the surface.

Example 3. *Consider the cylinder $x^2 + y^2 = 9$, $1 \le z \le 4$.*



From Ex 1, $\Phi_1(\theta, z) = (3\cos\theta, 3\sin\theta, z)$.

$T_\theta = (-3\sin\theta, 3\cos\theta, 0)$

$T_z = (0, 0, 1)$

$T_\theta \times T_z = \langle 3\cos\theta, 3\sin\theta, 0 \rangle$.

$\Phi_2(z, \theta) = (3\cos\theta, 3\sin\theta, z)$.

$T_z \times T_\theta = -(3\cos\theta, 3\sin\theta, 0)$

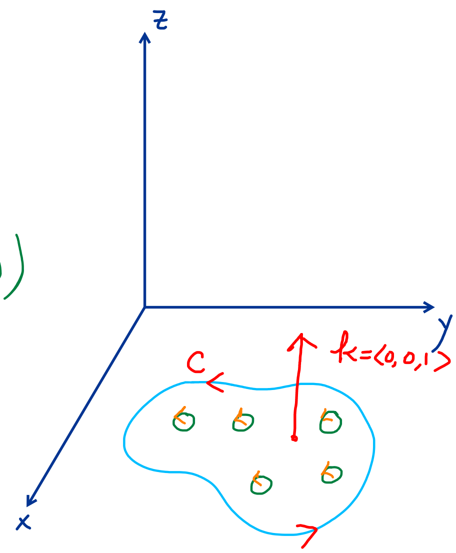
Fact. *Let S be an oriented surface.*

1. *Let F be a continuous vector field defined on S. Then*
   - *If Φ_1 and Φ_2 are two regular orientation-preserving parametrizations:*
     $$\int\int_{\Phi_1} F \cdot d\boldsymbol{S} = \int\int_{\Phi_2} F \cdot d\boldsymbol{S}$$
   - *If Φ_1 is orientation-preserving parametrization and Φ_2 is orientation-reversing parametrization:*
     $$\int\int_{\Phi_1} F \cdot d\boldsymbol{S} = -\int\int_{\Phi_2} F \cdot d\boldsymbol{S}$$
2. *If f is a real-valued function on S, and if Φ_1 and Φ_2 are parametrizations of S, then*
   $$\int\int_{\Phi_1} f dS = \int\int_{\Phi_2} f dS.$$

8.2 Stokes' Theorem

✓ Recall: "Green's theorem" applies only to **2-dimensional** vector fields F and **2-dimensional** region D,

$$\overbrace{\int_C F \cdot d\mathbf{s}}^{\text{"Circulation of } F \text{ around } C\text{"}} = \int\int_D \text{"microscopic circulation of } F\text{"}\, dA$$



2-dimensional region D.

Recall: Green's theorem: $F = \langle P(x,y), Q(x,y), 0 \rangle$.

C is $\overbrace{\text{the boundary of } D}^{\partial D}$, oriented counterclockwise.

microscopic circulation of F: $\operatorname{curl} F \cdot k$, $(k = (0,0,1))$

$$\int_C F \cdot d\mathbf{s} = \iint_D \operatorname{curl} F \cdot k \; dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

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§ Generalize to a surface S:

microscopic circulation on S = $\operatorname{curl} F \cdot n$

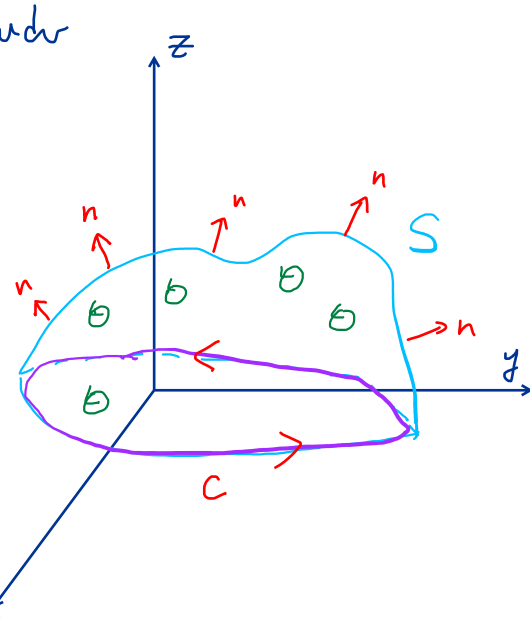
Total circulation

$$= \int_C F \cdot d\mathbf{s} = \iint_S (\operatorname{curl} F \cdot n)\, dS$$

$$= \iint \operatorname{curl} F \cdot \left(\frac{T_u \times T_v}{\cancel{\|T_u \times T_v\|}} \right) \cancel{\|T_u \times T_v\|}\, du\, dv$$

$$= \iint \operatorname{curl} F \cdot \underbrace{(T_u \times T_v)\, du\, dv}_{d\mathbf{S}}$$

$$= \iint \operatorname{curl} F \; d\mathbf{S}$$



Stokes' theorem generalizes Green's theorem to **3-dimensions**.

Fact. *(Stokes' Theorem) Let S be an oriented surface defined by a parametrization $\Phi : D \to S$, where D is a region in $\mathbb{R}^2$ to which Green's Theorem applies. Let C be the oriented boundary of S. Let F be a vector field on S. Then*

$$\int_C F \cdot d\mathbf{s} = \int \int_S curl F \cdot d\mathbf{S}.$$

Remark: In other words, Stokes' theorem relates the line integral of a vector field around a simple closed curve C to a surface integral for which C is surface's boundary.

For any surface S has **the same boundary** C, since

$$\text{the total circulation} \quad \int_C F \cdot d\mathbf{s} \quad \text{is equal to} \int\int_S \text{curl} F \cdot d\mathbf{S},$$

their surface integrals $\int\int_S \text{curl} F \cdot d\mathbf{S}$ must be the same.

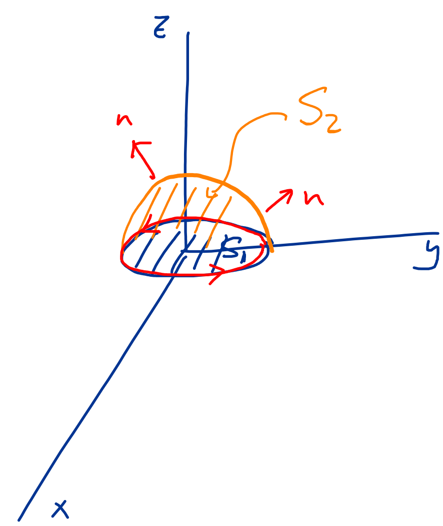
Example: Let C be unit circle $x^2 + y^2 = 1$, oriented counterclockwise viewed from positive z-axis.

Surface S_1 : $x^2 + y^2 \leq 1$, $z = 0$, unit disk.

S_1 has C as its boundary.

Surface S_2 : $x^2 + y^2 + z^2 = 1$, $z \geq 0$, upper unit sphere

Thus S_1, S_2 have the same boundary C.



NOTE that

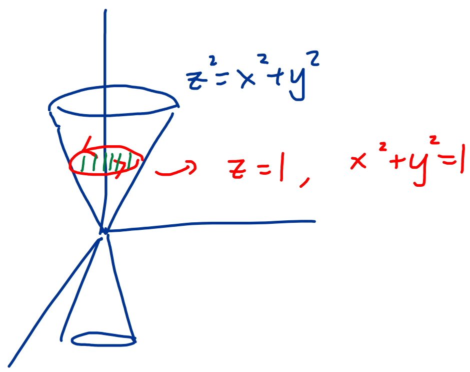
$$\iint_{S_1} (\nabla \times F) \cdot dS \overset{\text{Stokes'}}{=\!=} \int_C F \cdot ds$$

$$\iint_{S_2} (\nabla \times F) \cdot dS \overset{\text{Stokes'}}{=\!=} \int_C F \cdot ds$$

Since right hand sides of the above 2 identities are the same, one has

$$\iint_{S_1} (\nabla \times F) \cdot dS = \iint_{S_2} (\nabla \times F) \cdot dS.$$

Example 1. *Let* $F(x,y,z) = (\sin x - \frac{y^3}{3}, \cos y + \frac{x^3}{3}, xyz)$. *Compute* $\int_C F \cdot d\mathbf{s}$, *where* C *is the curve in which the cone* $z^2 = x^2 + y^2$ *intersects the plane* $z = 1$, *oriented counterclockwise when viewed from far out on the* $+z$*-axis.*



① Compute directly by definition of line integral.

$c(t) = (\cos t, \sin t, 1)$

$$\int F \cdot d\vec{s} = \int_0^{2\pi} F(c(t)) \cdot c'(t)\, dt$$

↪ It is not easy to compute.

② By Stokes' theorem,

$$\int_C F \cdot d\vec{s} = \iint_S (\nabla \times F) \cdot d\vec{S}$$

ⓐ $\nabla \times F = \langle xz, -yz, x^2 + y^2 \rangle$.

ⓑ $S : x^2 + y^2 \le 1, \ z = 1.$

$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, 1), \quad 0 \le r \le 1, \ 0 \le \theta < 2\pi$

$T_r = (\cos\theta, \sin\theta, 0)$

$T_\theta = (-r\sin\theta, r\cos\theta, 0)$

$T_r \times T_\theta = (0, 0, r)$, points upward.

$$\iint (\nabla \times F) \cdot d\vec{S} = \int_0^{2\pi}\int_0^1 (\nabla \times F)(\Phi(r,\theta)) \cdot (T_r \times T_\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r\cos\theta, -r\sin\theta, r^2) \cdot (0, 0, r)\, dr\, d\theta.$$

$$= \int_0^{2\pi}\int_0^1 r^3\, dr\, d\theta$$

$$= \pi/2. \ \#$$