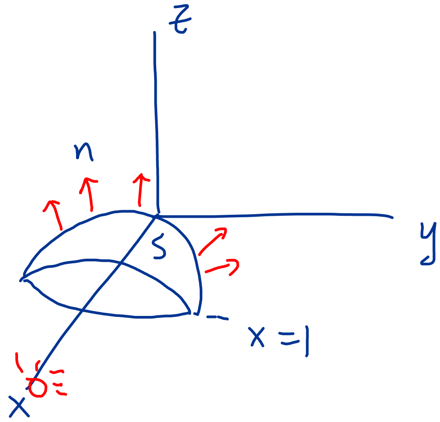
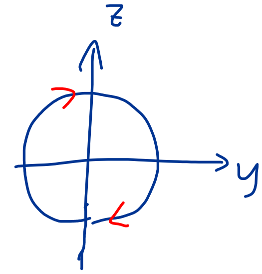
Quick Review from last week

- Stokes' Theorem:
$$\int_C F \cdot d\mathbf{s} = \int\int_S \text{curl} F \cdot d\mathbf{S},$$
where C be the oriented boundary of S.

Ex: $x = y^2 + z^2$, $x \leq 1$, normal is pointing $-x$ direction.

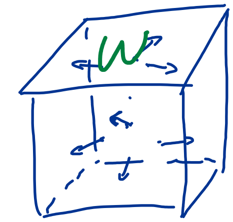




8.4 Gauss' Theorem

Suppose a vector field F represents the flow of a fluid. Recall the divergence of F ($\boxed{\text{div}F}$ or $\nabla \cdot F$) represents the "expansion or compression" of the fluid. gas

div$F > 0$ div$F < 0$.



Box.

The divergence(Gauss) theorem says that
"The total expansion of the fluid inside 3D region W" equals

$$\iiint_W \text{div}F \, dV$$

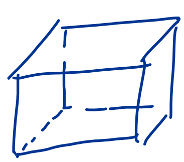
$$\|$$

"the total flux of the fluid out of the boundary of W"

$$\iint_{\partial W} F \cdot dS \qquad \partial W$$

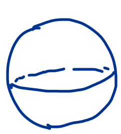
Definition:
Let W be an elementary region in $\mathbb{R}^3$. If the boundary of W is a surface made up of a finite number of surfaces, then we call the boundary of W is a **closed surface**.

Example 1. *1. Cube is an elementary region and its boundary is composed of* 6 *rectangles.*



2. Sphere is the boundary of a solid ball.

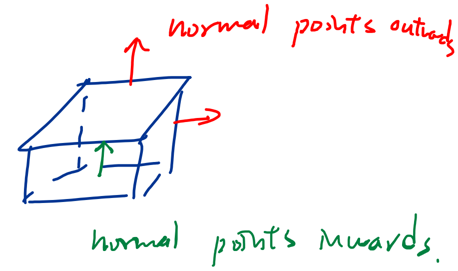
EX:



ball $x^2+y^2+z^2 \leq 4$.

sphere $x^2+y^2+z^2 = 4$.

Definition: Orientations in a **closed surface**:



- Outward pointing normal: normal points outwards.
- Inward pointing normal: normal points inwards.

Fact. *(Gauss' Theorem) or Divergence Theorem.*
Let W be an elementary region in $\mathbb{R}^3$ whose boundary ∂W, oriented with outward pointing normal. Let F be a smooth vector field on W. Then

$$\int\int_{\partial W} F \cdot d\boldsymbol{S} = \int\int\int_{W} (\nabla \cdot F) dV. \tag{1}$$

Example 2. *Let*

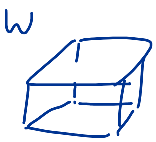
$$F(x,y,z) = (2x - z,\ x^2y,\ -xz^2).$$

Evaluate

$$\int\int_{\partial W} F \cdot d\boldsymbol{S},$$

where W is the unit cube $[0,1]\times[0,1]\times[0,1]$, $\boldsymbol{n}$ is the outward pointing normal.

Remark: If we compute $\int\int_{\partial W} F \cdot d\mathbf{S}$ directly by using the definition of surface integral, then we have to parametrize 6 boundary of W and compute them individually. Thus, for this problem, it is "much" easier to compute $\int\int_{\partial W} F \cdot d\mathbf{S}$ by using Gauss' Theorem (Divergence Theorem) than by computing it directly.

① 

, $\displaystyle\iint_{\partial W} F \cdot dS = \iint_{E_1} F \cdot dS + \cdots + \iint_{E_6} F \cdot dS$

$E_1, \dots, E_6$ are 6 boundary of W.

② Apply Gauss' theorem,

$$\iint_{\partial W} F \cdot dS = \iiint_W (\nabla \cdot F)\, dV$$

$$= \int_0^1\int_0^1\int_0^1 (\nabla \cdot F)\, dx\, dy\, dz$$

$$\nabla \cdot F = 2 + x^2 - 2xz$$

$$= \int_0^1\int_0^1\int_0^1 (2 + x^2 - 2xz)\, dx\, dy\, dz$$

$$= 11/6. \quad \#$$

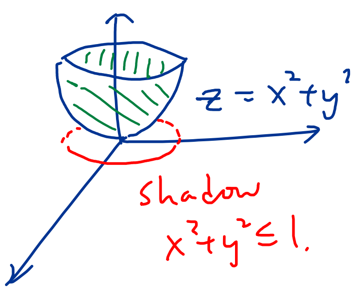
Example 3. *Consider a solid W bounded by $z = 1$ and $z = x^2 + y^2$, that is, W is described by $x^2 + y^2 \le z \le 1$. Let*

$$F(x, y, z) = (2x + z^2,\ x^5 + z^7,\ \cos(x^2) + \sin(y^3) - z^2).$$

Evaluate

$$\int\int_S F \cdot d\boldsymbol{S},$$

where S is the boundary of the solid W, $\boldsymbol{n}$ is the $\boxed{\text{outward pointing normal.}}$



By Gauss' theorem,

$$\iint_S F \cdot dS = \iiint_W (\nabla \cdot F)\, dV$$

$$\nabla \cdot F = 2 + 0 + (-2z)$$

$$= \iiint_W (2 - 2z)\, dV$$

Cylindrical coord.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\underbrace{x^2+y^2}_{r^2} \le z \le 1$$

$$0 \le r \le 1$$

$$0 \le \theta < 2\pi$$

$$= \int_0^{2\pi}\int_0^1\int_{r^2}^1 (2 - 2z)\, dz\ r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 2z - z^2 \Big|_{r^2}^1\ r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left[(2-1) - (2r^2 - r^4)\right] r\, dr\, d\theta$$

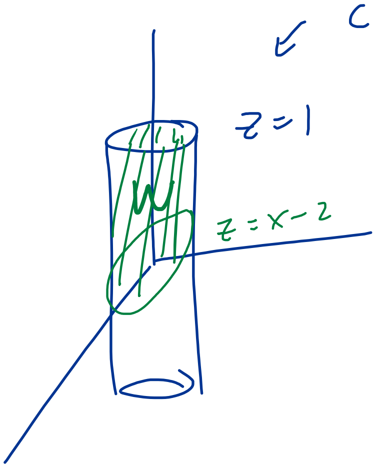
$$= \int_0^{2\pi}\int_0^1 (r - 2r^3 + r^5)\, dr\, d\theta$$

$$= \underline{\pi/3.}\ \#$$

Example 4. *Let $F = (xy^2,\ x^2y,\ y)$ and S is the surface of the cylinder $x^2+y^2=1$, bounded by the planes $z=1$ and $z=x-2$, and including the portions of $z=1$ and $z=x-2$ in the region $x^2+y^2\le 1$ with outward pointing normal. Evaluate*

S is closed surface

$$\iint_S F\cdot d\boldsymbol{S}.$$



By Gauss' theorem,

$$\iint F\cdot dS = \iiint_W (\nabla\cdot F)\,dV$$

$$= \iiint_W \underbrace{(x^2+y^2)}_{r^2}\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_{r\cos\theta-2}^1 r^2\,dz\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3\left(z\Big|_{r\cos\theta-2}^{1}\right)dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3(1-r\cos\theta+2)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 3r^3 - r^4\cos\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{3}{4}r^4 - \frac{1}{5}r^5\cos\theta\Big|_0^1\,d\theta$$

$$= \int_0^{2\pi} \frac{3}{4} - \frac{1}{5}\cos\theta\,d\theta$$

$$= \frac{3\pi}{2}.\ \#$$

Cylindrical coord.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z. \end{cases}$$

$0 \le r \le 1$

$0 \le \theta < 2\pi$

$\underbrace{x-2}_{r\cos\theta-2} \le z \le 1$