Quick Review from previous lecture

- Orientations in a **closed surface**:
  - **Outward pointing normal**: normal points outward.
  - **Inward pointing normal**: normal points inward.

- **The divergence (Gauss) theorem** says that

  Let $F$ be a smooth vector field on $W$. Then

  \[
  \int \int \int_W (\nabla \cdot F) dV = \int \int_{\partial W} F \cdot dS,
  \]

  where $W$ has boundary $\partial W$, oriented with outward pointing normal.

  This means
  “The total expansion of the fluid inside 3D region $W$” equals
  “the total flux of the fluid out of the boundary of $W$”

**Quiz 9**: 8.4, 3.1, 3.2.
3.1 Iterated partial derivatives

If \( f : \mathbb{R}^2 \to \mathbb{R}^1 \), then \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \) are also functions of two variables. The partial derivatives of \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \) are

\[
\begin{align*}
  f_{xx} &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right), \\
  f_{yy} &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right), \\
  f_{xy} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right), \\
  f_{yx} &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)
\end{align*}
\]

Example 1. Let \( f(x, y) = e^{xy} + y \cos(x) \). Find \( f_{xx}, f_{xy}, f_{yx}, f_{yy} \).

\[
\begin{align*}
  f_x &= y \ e^{xy} + (-y \sin x) \\
  f_y &= x \ e^{xy} + \cos x.
\end{align*}
\]

\[
\begin{align*}
  f_{xx} &= \frac{\partial}{\partial x} (f_x) = \frac{\partial}{\partial x} (y \ e^{xy} - y \sin x) = y^2 \ e^{xy} - y \cos x \\
  f_{xy} &= \frac{\partial}{\partial y} (f_x) = e^{xy} + xy \ e^{xy} - \sin x \\
  f_{yy} &= \frac{\partial}{\partial y} (f_y) = \frac{\partial}{\partial y} (x \ e^{xy} + \cos x) = x^2 \ e^{xy} + 0 \\
  f_{yx} &= \frac{\partial}{\partial x} (f_y) = \frac{\partial}{\partial x} (x \ e^{xy} + \cos x) = e^{xy} + xy \ e^{xy} - \sin x
\end{align*}
\]

We saw that \( f_{xy} = f_{yx} \).
3.2 Taylor’s Theorem

Recall “what is Linear approximation”: That is, we want to approximate $f(x)$ near $x = a$ by using a line. We take a line through the point $(a, f(a))$ with slope $f'(a)$:

$$T_1(x) = f(a) + f'(a)(x - a).$$

We call it the first order Taylor polynomial (or Linear approximation) of $f$ near $a$.

In Calculus 2, you also learned the second order Taylor polynomial (or quadratic approximation) of $f$ near $a$:

$$T_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2.$$  \hfill (2)

Now let’s start section 3.2.

We want to generalize the Taylor polynomial to functions of multiple variables.

**Fact.** (Taylor Polynomials for $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$)

We consider a $C^2$ function $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ with $n$ variables. Let $f$ be differentiable at $a$. Denote

$$x = (x_1, x_2, \ldots, x_n),$$
$$a = (a_1, a_2, \ldots, a_n).$$

- Then the first order Taylor polynomial (approximation) of $f$ at $a$ is

$$T_1(x) = f(a) + \text{matrix of partial derivatives} \cdot (x - a),$$

that is,

$$T_1(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i).$$
Then the second order Taylor polynomial (approximation) of \( f \) at \( a \) is

\[
T_2(x) = f(a) + \mathbf{D}f(a)(x - a) + \frac{1}{2!}(x - a)^T \mathbf{H}f(a)(x - a),
\]

that is,

\[
T_2(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j).
\]

\[\mathbf{D}f(a) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(a) & \cdots & \frac{\partial f}{\partial x_n}(a) \end{bmatrix}\]

\[\mathbf{D}f(a) \mathbf{x} - \mathbf{a} = \begin{bmatrix} \frac{\partial f}{\partial x_1}(a) & \cdots & \frac{\partial f}{\partial x_n}(a) \end{bmatrix} \begin{bmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{bmatrix}
= \frac{\partial f}{\partial x_1}(a)(x_1 - a_1) + \frac{\partial f}{\partial x_2}(a)(x_2 - a_2) + \cdots + \frac{\partial f}{\partial x_n}(a)(x_n - a_n).
\]

Hessian matrix of \( f \):

\[
\mathbf{H}f(a) = \begin{bmatrix}
\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\
\frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2}
\end{bmatrix}_{n \times n}
\]

\[(x - a)^T \mathbf{H}f(x - a) = \begin{bmatrix} x_1 - a_1 & \cdots & x_n - a_n \end{bmatrix} \begin{bmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{bmatrix}.
\]
Example 2. Find 1\textsuperscript{st} and 2\textsuperscript{nd} order Taylor approximation of 
\[ f(x, y) = 2x^2 + xy + 4y^2 - 1 \]
at the point \((x_0, y_0) = (1, 2)\).

\[ f(1, 2) = 19 \]
\[ f_x = 4x + y , \quad f_x(1,2) = 6 \]
\[ f_y = x + 8y , \quad f_y(1,2) = 17 \]
\[ f_{xx} = \frac{\partial}{\partial x} (4x + y) = 4. \]
\[ f_{yy} = \frac{\partial}{\partial y} (x + 8y) = 8. \]
\[ f_{xy} = \frac{\partial}{\partial y} (4x + y) = 1. \]

\[ T_1(x, y) = f(1,2) + f_x(1,2) (x-1) + f_y(1,2) (y-2). \]
\[ = 19 + 6 (x-1) + 17 (y-2). \]

\[ T_2(x, y) = f(1,2) + f_x(1,2) (x-1) + f_y(1,2) (y-2). \]
\[ + \frac{1}{2} \left[ f_{xx} (1,2) (x-1)^2 + f_{yy} (1,2) (y-2)^2 \right. \]
\[ \left. + 2 f_{xy} (1,2) (x-1) (y-2) \right]. \]
\[ = 19 + 6 (x-1) + 17 (y-2) + \frac{1}{2} \left[ 4(x-1)^2 + 8(y-2)^2 \right. \]
\[ \left. + 2 (x-1)(y-2) \right]. \]
Example 3. Consider the function \( f(x, y, z) = (x^2 + y^2 + z^2)^{1/2} \).

1. Find a linear approximation of \( f \) near \((4, 4, 2)\).

\[
f(4, 4, 2) = (4^2 + 4^2 + 2^2)^{1/2} = (36)^{1/2} = 6.
\]
\[
T_x = \frac{1}{2} \left( x^2 + y^2 + z^2 \right)^{-1/2} \cdot 2x, \quad T_x(4, 4, 2) = \frac{1}{2} \left( 16 + 16 + 4 \right)^{-1/2} \cdot 8 = \frac{1}{2} \cdot \frac{1}{6} \cdot 8.
\]
\[
T_y = \frac{1}{2} \left( x^2 + y^2 + z^2 \right)^{-1/2} \cdot 2y, \quad T_y(4, 4, 2) = \frac{1}{2} \left( 16 + 16 + 4 \right)^{-1/2} \cdot \frac{3}{2} = \frac{3}{3}.
\]
\[
T_z = \frac{1}{2} \left( x^2 + y^2 + z^2 \right)^{-1/2} \cdot 2z, \quad T_z(4, 4, 2) = \frac{1}{3}.
\]
\[
T_1(x, y, z) = 6 + \frac{2}{3} (x-4) + \frac{2}{3} (y-4) + \frac{1}{3} (z-2).
\]

2. Estimate the value \((4.01^2 + 3.99^2 + 2.03^2)^{1/2}\) by using the linear approximation you found in (a).

\[
\left( (4.01)^2 + (3.99)^2 + (2.03)^2 \right)^{1/2} \approx T_1(4.01, 3.99, 2.03) = 6 + \frac{2}{3} (0.01) + \frac{2}{3} (-0.01) + \frac{1}{3} (0.03) = 6.01.
\]

(approximated value)