Math 2374 Spring 2018 - Week 14

Quick Review from previous lecture

- Orientations in a **closed surface**:
  - Outward pointing normal: normal points outward.
  - Inward pointing normal: normal points inward.
- **The divergence (Gauss) theorem** says that
  Let F be a smooth vector field on W. Then

$$\int\int\int_W (\nabla \cdot F) dV = \overbrace{\int\int_{\partial W} F \cdot d\mathbf{S}}^{\text{flux}}, \tag{1}$$

  where W has boundary ∂W, oriented with outward pointing normal.

  This means
  "The total expansion of the fluid inside 3D region W" equals
  "the total flux of the fluid out of the boundary of W"

Quiz 9: 8.4, 3.1, 3.2.

3.1 Iterated partial derivatives

$f(x,y)$ scalar valued.

If $f : \mathbb{R}^2 \to \mathbb{R}^1$, then $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are also functions of two variables. The partial derivatives of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$
$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Example 1. *Let $f(x,y) = e^{xy} + y\cos(x)$. Find $f_{xx}, f_{xy}, f_{yx}, f_{yy}$.*

$$f_x = y\,e^{xy} + (-y\sin x)$$

$$f_y = x\,e^{xy} + \cos x.$$

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(y\,e^{xy} - y\sin x\right) = y^2 e^{xy} - y\cos x.$$

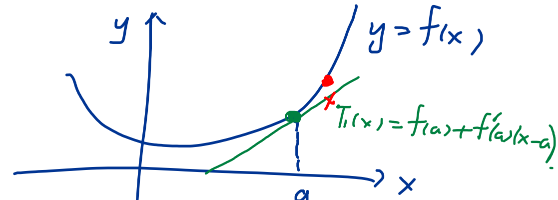
$$\underline{f_{xy}} = \frac{\partial}{\partial y}(f_x) = \underline{e^{xy} + xy\,e^{xy} - \sin x}.$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(x\,e^{xy} + \cos x\right) = x^2 e^{xy} + 0.$$

$$\underline{f_{yx}} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(x\,e^{xy} + \cos x\right) = \underline{e^{xy} + xy\,e^{xy} - \sin x}.$$

We saw that $f_{xy} = f_{yx}$.

3.2 Taylor's Theorem



Recall "what is Linear approximation":

1 variable.

That is, we want to approximate $f(x)$ near $x = a$ by using a line. We take a line through the point $(a, f(a))$ with slope $f'(a)$:

$$T_1(x) = f(a) + f'(a)(x - a).$$

We call it the **first order Taylor polynomial**(or Linear approximation) of f near a.

(approxmation)

In Calculus 2, you also learned the **second order Taylor polynomial** (or quadratic approximation) of f near a:

$$T_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2. \tag{2}$$

n-th Taylor polynomial $T_n(x) = \sim\sim\sim + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n.$

Now let's start section 3.2.

We want to generalize the Taylor polynomial to functions of multiple variables.

Fact. *(Taylor Polynomials for $f : \mathbb{R}^n \to \mathbb{R}^1$)*
We consider a C^2 function $f : \mathbb{R}^n \to \mathbb{R}^1$ with n variables. Let f be differentiable at a. Denote

$$x = (x_1, x_2, \cdots, x_n),$$
$$a = (a_1, a_2, \cdots, a_n).$$

- *Then the* ***first order Taylor polynomial (approximation)*** *of f at a is*

  $$T_1(x) = f(a) + \overbrace{\mathbf{D}f(a)}^{\text{matrix of partial derivatives}}(x - a),$$

  that is,

  $$T_1(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i).$$

- *Then the* ***second order Taylor polynomial (approximation)*** *of f at a is*

$$T_2(x) = f(a) + \mathbf{D}f(a)(x-a) + \frac{1}{2!}(x-a)^T\mathbf{H}f(a)(x-a),$$

that is,

$$T_2(x) = f(a) + \boxed{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i)} + \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j).$$

$$Df(a) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(a) & \cdots & \frac{\partial f}{\partial x_n}(a) \end{bmatrix}$$

$$Df(a)(x-a) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(a) & \cdots & \frac{\partial f}{\partial x_n}(a) \end{bmatrix} \begin{bmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{bmatrix}$$

$$= \frac{\partial f}{\partial x_1}(a)(x_1 - a_1) + \frac{\partial f}{\partial x_2}(a)(x_2 - a_2) + \cdots + \frac{\partial f}{\partial x_n}(a)(x_n - a_n).$$

Hessian matrix of f:

$$Hf(a) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}_{n \times n}$$

$$(x-a)^T Hf(x-a) = \begin{bmatrix} x_1 - a_1 & \cdots & x_n - a_n \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{bmatrix}.$$

Example 2. *Find 1^{st} and 2^{nd} order Taylor approximation of*

$$f(x,y) = 2x^2 + xy + 4y^2 - 1$$

at the point $(x_0, y_0) = (1, 2)$.

$f(1,2) = 19$

$f_x = 4x + y$, $f_x(1,2) = 6$

$f_y = x + 8y$, $f_y(1,2) = 17$

$f_{xx} = \frac{\partial}{\partial x}(4x + y) = 4.$

$f_{yy} = \frac{\partial}{\partial y}(x + 8y) = 8.$

$f_{xy} = \frac{\partial}{\partial y}(4x + y) = 1.$

$T_1(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2).$

$= 19 + 6(x-1) + 17(y-2).$ #

$T_2(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2).$

$+ \frac{1}{2}\Big[f_{xx}(1,2)(x-1)^2 + f_{yy}(1,2)(y-2)^2$

$+ 2f_{xy}(1,2)(x-1)(y-2) \Big].$

$= 19 + 6(x-1) + 17(y-2) + \frac{1}{2}\Big[4(x-1)^2 + 8(y-2)^2$

$+ 2(x-1)(y-2)\Big]$

#

Example 3. *Consider the function* $f(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$.

(1st Taylor polynomial)

1. *Find a linear approximation of* f *near* $(4, 4, 2)$.

$f(4,4,2) = (4^2+4^2+2^2)^{1/2} = (36)^{1/2} = 6.$

$f_x = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\, 2x, \quad f_x(4,4,2) = \frac{1}{2}(16+16+4)^{-1/2}\, 8 = \frac{1}{2}\frac{1}{6}8 = \frac{2}{3}.$

$f_y = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\, 2y, \quad f_y(4,4,2) = \frac{1}{2}(16+16+4)^{-1/2}\, 8 = \frac{2}{3}$

$f_z = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\, 2z, \quad f_z(4,4,2) = 1/3.$

$T_1(x,y,z) = 6 + \frac{2}{3}(x-4) + \frac{2}{3}(y-4) + \frac{1}{3}(z-2)$. #

2. *Estimate the value* $(4.01^2 + 3.99^2 + 2.03^2)^{1/2}$ *by using the linear approximation you found in* (*a*).

real value (calculator : we get 6.01008....)

$\left((4.01)^2 + (3.99)^2 + (2.03)^2\right)^{1/2}$

$\sim T_1(4.01, 3.99, 2.03) = 6 + \cancel{\frac{2}{3}(0.01) + \frac{2}{3}(-0.01)} + \frac{1}{3}(0.03)$

$= 6.01.$

(approximated value)