Quick Review from previous lecture

- The **first order Taylor approximation** (Linear approximation) of f near a:

$$T_1(x) = f(a) + \mathbf{D}f(a)(x-a), \quad \mathbf{D}f = [f_{x_1} \quad \cdots \quad f_{x_n}].$$

or

$$T_1(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i).$$

- The **second order Taylor approximation** (Quadratic approximation) of f near a:

$$T_2(x) = f(a) + \mathbf{D}f(a)(x-a) + \frac{1}{2!}(x-a)^T \mathbf{H}f(a)(x-a),$$

or

$$T_2(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j).$$

Hessian matrix of f:

$$Hf(a) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}_{n \times n}$$

Quiz 10: 3.3

3.3 Extrema of real-valued functions

Recall: Extrema for one-variable function.

How to find extrema for a function with 1 variable? in Cal 1.

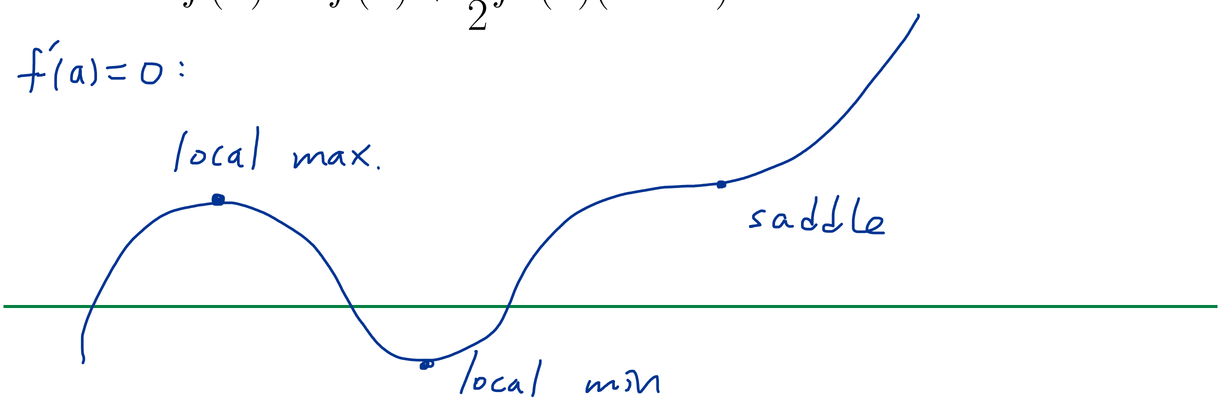
1st step is to find the critical points:

We call $x = a$ is a critical point of $f(x)$ if $f'(a) = 0$ or if $f'(a)$ is not defined.

To decide if f has a local min or local max at a critical point $x = a$, then we should look at **2nd order Taylor polynomial**:

$$f(x) \approx f(a) + \frac{1}{2} f''(a)(x-a)^2.$$

Suppose $f'(a) = 0$:



- $f''(a) > 0$: $T_2(x) = f(a) + \frac{1}{2} f''(a)(x-a)^2$, parabola.

  f is approximated by a parabola pointing upwards so f has a minimum at $x = a$.

- $f''(a) < 0$:

  f is approximated by a parabola pointing downwards so f has a maximum at $x = a$.

- $f''(a) = 0$:

  NO information.

Now in section 3.3, we will discuss how to determine the extrema of a function with two variables.

§Extrema for two-variable functions.

Suppose $f(x, y)$ is differentiable. The local extrema can occur only at critical points $(x, y) = (a, b)$, that is, the matrix of partial derivatives of f at $(x, y) = (a, b)$ is

$\to f_x(a,b) = 0, \; f_y(a,b) = 0.$

$$\mathbf{D}f(a, b) = [0\ 0], \text{ a } 1 \times 2 \text{ matrix.}$$

If $\mathbf{D}f(a, b) = [0\ 0]$, then we look at its second order Taylor polynomial

$$T_2(x, y) = f(a, b) + \frac{1}{2}[x - a \quad y - b]\mathbf{H}f(a, b) \begin{bmatrix} x - a \\ y - b \end{bmatrix}$$

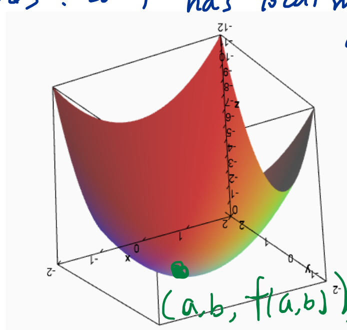
$f(x,y)$

$$Hf(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{2\times 2}$$

- $\mathbf{H}f(a, b)$ is **positive definite**,
  then the graph of $f(x, y)$ looks like elliptic paraboloid pointing upwards. So f has local min. at (a,b)

  $f_{xx}(a,b) > 0,$

  $\det Hf(a,b) > 0.$

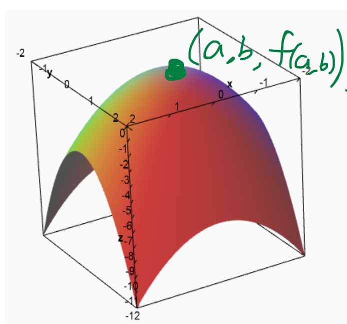


$(a,b, f(a,b))$

- $\mathbf{H}f(a, b)$ is **negative definite**,
  then the graph of $f(x, y)$ looks like elliptic paraboloid pointing downwards, f has local max. at (a,b)

  $f_{xx}(a,b) < 0.$

  $\det Hf(a,b) > 0.$



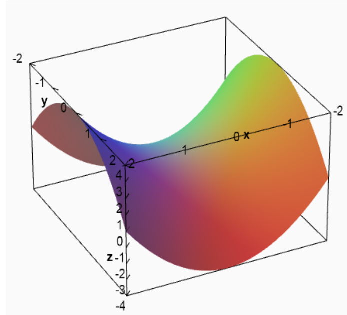
$(a,b, f(a,b))$

$\det Hf(a,b) < 0.$

- $\mathbf{H}f(a, b)$ is **indefinite**, then the graph of $f(x, y)$ looks like hyperbolic paraboloid, and f has neither a local maximum nor a local minimum at the critical point.

  Note that a critical point that is not an extrema, then it is called a **saddle point**.

All pictures are taken from math insight.



§How do we find local max. and min.

Second derivative test:

1. Find all critical points: Find (a, b) such that

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0.$$

2. Denote $Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$

$$det(\mathbf{H}f) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = f_{xx}f_{yy} - (f_{xy})^2.$$

- If $det(\mathbf{H}f) = 0$ at (a, b), then no information.
- If $det(\mathbf{H}f) < 0$ at (a, b), then (a, b) is saddle point.
- If $det(\mathbf{H}f) > 0$,

  (1) $\frac{\partial^2 f}{\partial x^2}(a,b) > 0$, then (a, b) is local minimum of f.

  (2) $\frac{\partial^2 f}{\partial x^2}(a,b) < 0$, then (a, b) is local maximum of f.

Example 1. *Classify each critical point of* $f(x, y) = x^3 + x^2y - y^2 - 4y$ *as a local maximum, local minimum, or saddle point.*

1. Find critical points.

$f_x = \boxed{3x^2 + 2xy} = 0 \Rightarrow x(3x+2y)=0$

$f_y = \boxed{x^2 - 2y - 4} = 0 \cdot$ ⊛ $\Rightarrow x = 0, \ (x = -\frac{2y}{3})$

$y = -\frac{3x}{2}$.

plug in ⊛

$\Rightarrow x^2 - 2\left(-\frac{3x}{2}\right) - 4 = 0$

$\Rightarrow x^2 + 3x - 4 = 0.$

$\Rightarrow (x-1)(x+4) = 0$

$\Rightarrow x = 1, -4$

$x = 0$, plug in to ⊛, $0 - 2y - 4 = 0$

$y = -2.$

$(0, -2).$

$x = 1, \quad y = -\frac{3x}{2} \Rightarrow y = -\frac{3}{2}. \quad (1, -3/2)$

$x = -4, \quad y = -\frac{3x}{2} \Rightarrow y = 6, \quad (-4, 6)$

2. Find Hf.

$f_{xx} = \frac{\partial}{\partial x}\left(\boxed{3x^2 + 2xy}\right) = 6x + 2y.$

$f_{yy} = \frac{\partial}{\partial y}\left(\boxed{x^2 - 2y - 4}\right) = -2.$

$f_{xy} = 2x$

$$Hf = \begin{bmatrix} \boxed{6x+2y} & 2x \\ 2x & -2 \end{bmatrix}.$$

① $(0, -2)$: local max. at $(0, -2)$

$f_{xx}(0, -2) = 0 - 4 < 0.$

$$\det Hf(0,-2) = \det \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix} > 0.$$

② $\underline{(1, -3/2)}$: saddle point

$$f_{xx}(1, -3/2) = 6 + 2(-3/2) = 6-3 > 0.$$

$$\det Hf(1,-3/2) = \begin{bmatrix} 3 & 2 \\ 2 & -2 \end{bmatrix} = -6\ \ 4 < 0.$$

③ $\underline{(-4, 6)}$: saddle point

$$\det Hf(-4, 6) = \begin{bmatrix} -24+12 & -8 \\ -8 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -8 \\ -8 & -2 \end{bmatrix} < 0.$$

Second derivative test in 3 variables. Consider a function $f(x, y, z)$.

1. Find all critical points:

$$f_x(a, b, c) = 0, \; f_y(a, b, c) = 0, \; f_z(a, b, c) = 0.$$

2. Find the Hessian matrix of f

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

Let

$$D_1 = f_{xx}, \quad D_2 = det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}, \quad D_3 = det \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}.$$

- If $D_1(a, b, c) > 0$, $D_2(a, b, c) > 0$, and $D_3(a, b, c) > 0$, then f has a local minimum at (a, b, c).
- If $D_1(a, b, c) < 0$, $D_2(a, b, c) > 0$, and $D_3(a, b, c) < 0$, then f has a local maximum at (a, b, c).
- In any other case where $D_3(a, b, c) \neq 0$, f has a saddle point at (a, b, c).

Example 2. *Classify each critical point of* $f(x,y,z) = x^3 + y^3 + z^2 \overset{-9x}{\wedge} - 9y - 4z$ *as a local maximum, local minimum, or saddle point.*

1. Find critical point:

$$f_x = 3x^2 - 9 \Rightarrow x = \pm\sqrt{3}.$$

$$f_y = 3y^2 - 9 \Rightarrow y = \pm\sqrt{3}$$

$$f_z = 2z - 4 \Rightarrow z = 2.$$

$(\pm\sqrt{3}, \pm\sqrt{3}, 2)$: 4 critical points.

2. Find Hf:

$$Hf = \begin{bmatrix} 6x & 0 & 0 \\ 0 & 6y & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{cases} f_{xx} = 6x \\ f_{yy} = 6y \\ f_{zz} = 2. \end{cases}$$

① $\underline{(\sqrt{3}, \sqrt{3}, 2)}$: $f_{xx}(\sqrt{3},\sqrt{3},2) = 6\sqrt{3} > 0.$

$$D_2(\sqrt{3},\sqrt{3},2) = \det\begin{bmatrix} 6\sqrt{3} & 0 \\ 0 & 6\sqrt{3} \end{bmatrix} > 0.$$

$$D_3(\sqrt{3},\sqrt{3},2) = \det\begin{bmatrix} 6\sqrt{3} & 0 & 0 \\ 0 & 6\sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix} > 0.$$

$(\sqrt{3},\sqrt{3},2)$: local min.

② $\underline{(\sqrt{3}, -\sqrt{3}, 2)}$: $f_{xx}(\sqrt{3}, -\sqrt{3}, 2) > 0.$

$$D_2(\sqrt{3}, -\sqrt{3}, 2) = \det\begin{bmatrix} 6\sqrt{3} & 0 \\ 0 & -6\sqrt{3} \end{bmatrix} < 0.$$

$D_3 \neq 0.$

$(\sqrt{3}, -\sqrt{3}, 2)$ saddle point.

③ $(-\sqrt{3}, \sqrt{3}, 2)$: $f_{xx}(-\sqrt{3}, \sqrt{3}, 2) < 0,$, saddle point.

$D_2 < 0.$ $D_3 \neq 0$

④ $(-\sqrt{3}, -\sqrt{3}, 2)$: $f_{xx}(-\sqrt{3}, -\sqrt{3}, 2) < 0$,

$D_2(-\sqrt{3}, -\sqrt{3}, 2) > 0$.

$$D_3 = \begin{bmatrix} -6\sqrt{3} & & \\ & -6\sqrt{3} & \\ & & 2 \end{bmatrix} > 0.$$

It is saddle point.

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