Example 1. *1. Parametrize*

(a) *The graph of a function $z = g(x,y)$.*

$\Phi(x,y) = (x, y, g(x,y))$.

(b) *Sphere $x^2 + y^2 + z^2 = 4$ and sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = 9$*

① $\Phi(\theta,\varphi) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)$, $0 \le \varphi \le \pi$, $0 \le \theta < 2\pi$.

② $\Phi(\theta,\varphi) = (a + 2\sin\varphi\cos\theta, b + 2\sin\varphi\sin\theta, c + 2\cos\varphi)$.

(c) *Ellipsoid $x^2 + 2y^2 + 3z^2 = 4$.*

$$\frac{x^2}{4} + \frac{y^2}{2} + \frac{z^2}{\frac{4}{3}} = 1.$$

$\Phi(\theta,\varphi) = (2\sin\varphi\cos\theta, \sqrt{2}\sin\varphi\sin\theta, \sqrt{\frac{4}{3}}\cos\varphi)$, $0 \le \varphi \le \pi$, $0 \le \theta < 2\pi$.

(d) *Cone $z = \sqrt{x^2+y^2}$.*

① $\Phi(x,y) = (x, y, \sqrt{x^2+y^2})$.

② cylindrical coord: $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r)$.

(e) *Paraboloid $z = x^2 + y^2$.*

① $\Phi(x,y) = (x, y, x^2+y^2)$

② $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$.

(f) *Plane $x + 2y + 3z = 1$.*

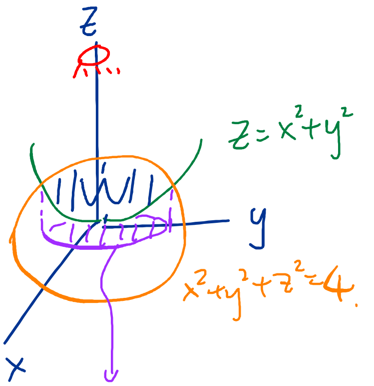
Use (a), $\Phi(x,y) = (x, y, \frac{1-x-2y}{3})$.

$\Phi(x,z) = (x, \frac{1-x-3z}{2}, z)$

$\Phi(y,z) = (1-2y-3z, y, z)$

2. *Find volume of a bounded solid W enclosed by the following two surfaces in part 1.*

- *W is bounded below by paraboloid* $z = x^2+y^2$ *and bounded above by sphere.* $x^2+y^2+z^2=4$.



$$x^2+y^2 \le z \le \sqrt{4-x^2-y^2}$$

Find intersection: $\begin{cases} z=x^2+y^2 \\ \underbrace{x^2+y^2}_{=z}+z^2=4 \end{cases} \Rightarrow z+z^2=4 \Rightarrow z^2+z-4=0 \Rightarrow z=\frac{-1\pm\sqrt{17}}{2}$.

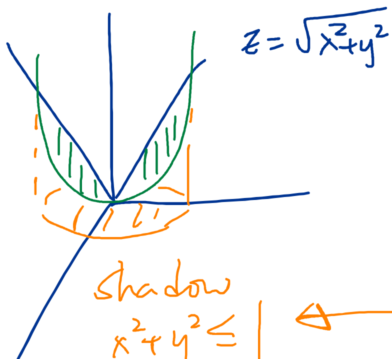
$$\boxed{z=\frac{-1+\sqrt{17}}{2}}$$

$$\iint_D \int_{x^2+y^2}^{\sqrt{4-x^2-y^2}} 1 \; dz\, dx dy$$

shadow

$D: x^2+y^2 \le \frac{-1+\sqrt{17}}{2}$.

- *W is bounded below by paraboloid and bounded above by cone*



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$x^2+y^2 \le 1$.

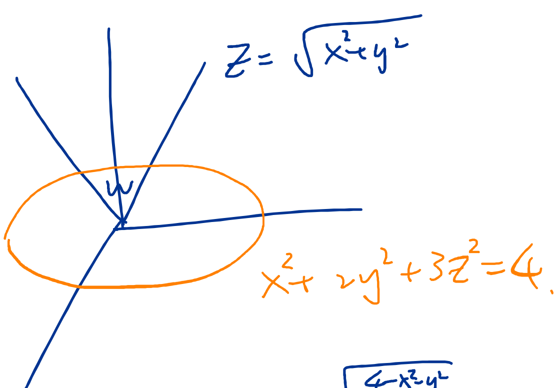
$$x^2+y^2 \le z \le \sqrt{x^2+y^2}.$$

Find intersection: $\begin{matrix} z=x^2+y^2 \\ z=\sqrt{x^2+y^2}. \end{matrix} \Rightarrow z=z^2. \Rightarrow z=0, 1.$

$$\iint_{x^2+y^2\le 1} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} 1 \; dz\, dx dy.$$

[To compute it, you can use cylindrical coord.]

- *W is bounded below by cone and bounded above by ellipsoid.*



$$\sqrt{x^2+y^2} \le z \le \sqrt{\frac{4-x^2-2y^2}{3}}.$$

Find intersection:

$\begin{cases} z=\sqrt{x^2+y^2} \\ x^2+2y^2+3z^2=4. \end{cases}$

$\Rightarrow x^2+2y^2+3(x^2+y^2)=4 \Rightarrow 4x^2+5y^2=4.$

$$\iint_{4x^2+5y^2\le 4} \int_{\sqrt{x^2+y^2}}^{\sqrt{\frac{4-x^2-y^2}{3}}} 1 \; dz\, dx dy \underset{\substack{z=z \\ x=r\cos\theta \\ y=\sqrt{\frac{4}{5}}\, r\sin\theta}}{=} \boxed{}$$

$0\le r\le 1$, $0\le\theta\le 2\pi$.

3. *Find the surface area of one surface that is cut off by other surface.*

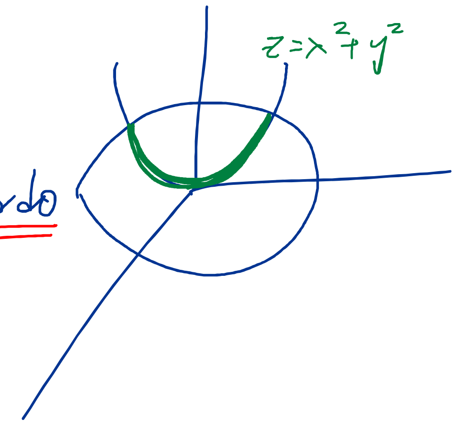
- *Find the area of paraboloid $z = x^2 + y^2$ that is inside the sphere $x^2 + y^2 + z^2 = 4$.*

Surface area $= \iint \|T_u \times T_v\| \, du \, dv = \iint \sqrt{1 + 4x^2 + 4y^2} \, dx \, dy$

$x^2 + y^2 = \frac{-1+\sqrt{17}}{2}$

polar coord $\int_0^{2\pi} \int_0^{\sqrt{\frac{-1+\sqrt{17}}{2}}} \sqrt{1+4r^2} \, r \, dr \, d\theta$

$\vdots$



Parameterization:

$\Phi(x, y) = (x, y, x^2 + y^2)$

$T_x = (1, 0, 2x)$

$T_y = (0, 1, 2y)$

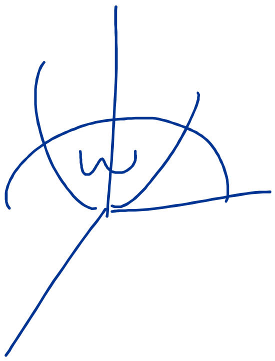
$T_x \times T_y = (-2x, -2y, 1)$

- *(HW) Find the area of surface A that is inside (or cut off by) the surface B. (chose surface A and B from part 1.)*

Example 2. *Evaluate the surface integral* $\int\int_S F \cdot d\boldsymbol{S}$, *where*

$$F(x, y, z) = (2xy,\ y^2 z,\ z)$$

and S *is the boundary of the solid* W *in Example 1, part 2, that is,* W *is bounded below by paraboloid* $z = x^2 + y^2$ *and bounded above by sphere* $x^2 + y^2 + z^2 = 4$. *with outward pointing normal.*



By Gauss' theorem,

$$\iint_S F \cdot d\vec{S} = \iiint_W \operatorname{div} F \, dV.$$

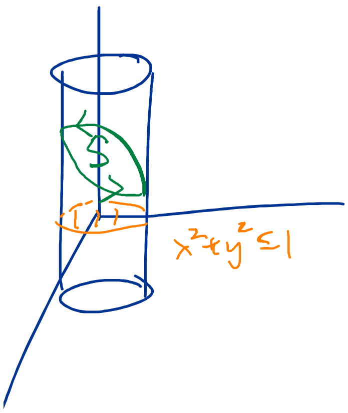
$$= \iiint_W (2y + 2yz + 1)\, dV.$$

copy bounds from Ex 1.

Example 3. *Use Stokes' Theorem to evaluate the line integral*

$$\int_C -y^3dx + x^3dy - z^3dz, \quad = \int F \cdot ds \quad , \quad F = (-y^3, x^3, -z^3).$$

where C is the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$, and the orientation on C corresponds to counterclockwise motion in the xy plane.



By Stokes' theorem,

$$\int_C F \cdot ds = \iint_S (\nabla \times F) \cdot dS = \iint_{x^2+y^2\le 1} (0, 0, 3x^2+3y^2)\cdot(1,1,1)\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 (3r^2)\, r\,dr\,d\theta.$$

$$= \frac{3\pi}{2}.$$

$S: \Phi(x,y) = (x, y, 1-x-y)$

$T_x = (1, 0, -1)$

$T_y = (0, 1, -1)$

$T_x \times T_y = (1, 1, 1)$

$\nabla \times F = (0, 0, 3x^2+3y^2).$

Example 4. *Find the critical points of the given function*

$$f(x,y) = x^3 + \frac{3}{2}x^2 - 6x + y^2 - 4y = 0.$$

and determine wheter they are local maxima, local minima, or saddle points.

$f_x = 3x^2 + 3x - 6 = 0 \quad \Rightarrow \quad x = 1, -2$

$f_y = 2y - 4 = 0 \quad \Rightarrow \quad y = 2.$

Critical points : $(1, 2)$, $(-2, 2)$.

Find Hf: $f_{xx} = 6x+3$, $f_{yy} = 2$ $f_{xy} = 0$.

$$Hf = \begin{bmatrix} 6x+3 & 0 \\ 0 & 2 \end{bmatrix}$$

① $\underline{(1,2)}$: $Hf(1,2) = \begin{bmatrix} 9 & 0 \\ 0 & 2 \end{bmatrix} > 0$. $f_{xx}(1,2) > 0$.

$(1,2)$ is local min.

② $(-2,2)$: $Hf(-2,2) = \begin{bmatrix} -12+3 & 0 \\ 0 & 2 \end{bmatrix} < 0$.

$(-2,2)$ is saddle point.