Math 2374

Final review 2

This review sheet is not meant to be your only form of studying. Understanding all the homework problems and lecture material are essential for success in the course. This review sheet only contains the key ideas of these sections.

- Let $T : D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping. Then T transforms parallelograms into parallelograms and vertices into vertices. Moreover, if $T(D^*)$ is a parallelogram, D^* must be a parallelogram.
- A mapping $T : D^* \to D$ is one-to-one when it maps distinct points to distinct points. It is onto when the image of D^* under T is all of D.
- A linear transformation of $\mathbb{R}^n$ to $\mathbb{R}^n$ given by multiplication by a matrix A is one-to-one and onto when and only when $\det A \neq 0$.
- Change of variables:
  1. $$\iint_D f(x,y)\,dxdy = \iint_{D^*} f\left(x(u,v), y(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$
     Polar coordinates: $x = r\cos(\theta)$ and $y = r\sin(\theta)$, then $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$.
  2. $$\iiint_W f(x,y,z)\,dxdydz = \iiint_{W^*} f\left(x(u,v,w), y(u,v,w), z(u,v,w)\right)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$
     Cylindrical coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$, then $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$.
     Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$
     with $0 \leq \rho,\ 0 \leq \phi \leq \pi,\ 0 \leq \theta < 2\pi$, then $\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2\sin(\phi)$.
- Parametrize key surfaces: spheres, cylinders, cones, planes, surface of form $z = g(x,y)$.
- Tangent vectors: $\mathbf{T}_u = \frac{\partial \mathbf{\Phi}}{\partial u}$, $\mathbf{T}_v = \frac{\partial \mathbf{\Phi}}{\partial v}$.
- Normal vector is $\mathbf{T}_u \times \mathbf{T}_v$.
- The surface S is called regular if $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$.
- The unit normal is $\mathbf{n} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{||T_u \times T_v||}$. Positive side of surface is the side with normal $\mathbf{n}$.
- If $\mathbf{T}_u \times \mathbf{T}_v \neq 0$, then the tangent plane of the surface at $\mathbf{\Phi}(u_0, v_0) = (x_0, y_0, z_0)$ is
  $$(\mathbf{T}_u \times \mathbf{T}_v)(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$
- Area$(S) = \iint_D ||\mathbf{T}_u \times \mathbf{T}_v||\, dudv$.
- If $\mathbf{\Phi}(u,v) = (u, v, g(u,v))$, then
  1. $\mathbf{T}_u \times \mathbf{T}_v = \left(-\frac{\partial g}{\partial u}, -\frac{\partial g}{\partial v}, 1\right)$;
  2. $||\mathbf{T}_u \times \mathbf{T}_v|| = \sqrt{\left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2 + 1}$.

- Surface integrals of scalar-valued function f:
$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{\Phi}(u,v))||\mathbf{T}_u \times \mathbf{T}_v||\,dudv$

- If the surface S is the graph of $g(u,v)$, then a parametrization of S is $\mathbf{\Phi}(u,v) = (u,v,g(u,v))$. We have $\iint_S f(x,y,z)\,dS = \iint_D f(u,v,g(u,v))\sqrt{1+\left(\frac{\partial g}{\partial u}\right)^2+\left(\frac{\partial g}{\partial v}\right)^2}\,dudv$

- Surface integrals of vector-valued function F:
$\iint_S \mathbf{F}\cdot d\mathbf{S} = \pm\iint_D \mathbf{F}(\mathbf{\Phi}(u,v))\cdot(\mathbf{T}_u \times \mathbf{T}_v)\ dudv.$
(minus sign if $T_u \times T_v$ points in the opposite direction as $\mathbf{n}$).

- If the surface S is the graph of $g(u,v)$, then a parametrization of S is $\mathbf{\Phi}(u,v) = (u,v,g(u,v))$. We have $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(u,v,g(u,v))\cdot\left(-\frac{\partial g}{\partial u}, -\frac{\partial g}{\partial v}, 1\right)\,dudv$

- Let $\mathbf{F} = (P,Q)$. Green's theorem is

$$\int_{\partial D} Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

  Here we need a "positively oriented" boundary $C = \partial D$ correctly. The region D must be on your left as you move along C.

- $\text{Area}(D) = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx.$

- Stokes' Theorem: $\iint_S (\nabla \times \mathbf{F})\cdot d\mathbf{S} = \int_{\partial S} \mathbf{F}\cdot d\mathbf{s}.$

- Key idea 1: To calculate $\int_C \mathbf{F}\cdot d\mathbf{s}$, you can choose any surface with boundary C and calculate $\iint_S (\nabla \times \mathbf{F})\cdot d\mathbf{S}$.

- Key idea 2: To calculate $\iint_S (\nabla \times \mathbf{F})\cdot d\mathbf{S}$: you can either
  1. convert it to the integral of $\mathbf{F}$ over the boundary ∂S, or
  2. change the surface S to any other surface S' with the same boundary $\partial S' = \partial S$ and compute the integral over S' rather than over S.

- Need positively oriented boundary, that is, when you walk on the positive side of surface near boundary and surface is on your left.

- Let $\mathbf{F}$ be a C^1 vector field that is defined on $\mathbb{R}^3$, except possibly for a finite number of points. The following conditions on $\mathbf{F}$ are equivalent:
  1. For any oriented simple closed curve C, $\int_C \mathbf{F}\cdot d\mathbf{s} = 0$.

2. For any two oriented simple curves C_1 and C_2 that have the same endpoints,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}.$$

3. $\mathbf{F}$ is the gradient of some function f.
4. $\nabla \times \mathbf{F} = \mathbf{0}$.

- $\mathbf{F}$ is a C^1 vector field on $\mathbb{R}^2$ of the form $P\mathbf{i} + Q\mathbf{j}$ that satisfies $\partial P/\partial y = \partial Q/\partial x$, then $\mathbf{F} = \nabla f$ for some f on $\mathbb{R}^2$.
- If $\mathbf{F}$ is a C^1 vector field on all of $\mathbb{R}^3$ with div $\mathbf{F} = 0$, then there exists a C^1 vector field $\mathbf{G}$ with $\mathbf{F} = \text{curl } \mathbf{G}$.
- First order Taylor approximation (Linear approximation):
  $L_1(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0)$,
  Second order Taylor approximation (Quadratic approximation):
  $L_2(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0) + \frac{1}{2} \sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)$.
- **Second derivative test for a function with 2 variables.**
  Let $f(x, y)$ be of class C^2 on an open set $U \subset \mathbb{R}^2$. Let (x_0, y_0) be a critical point of $f(x, y)$. Denote

$$det(\mathbf{H}f) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

  If $det(\mathbf{H}f)(x_0, y_0) > 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum.
  If $det(\mathbf{H}f)(x_0, y_0) > 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum.
  If $det(\mathbf{H}f)(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point.

- **Second derivative test for a function with 3 variables.**
  Consider a function $f(x, y, z)$.

  1. Find all critical points:

$$f_x(a, b, c) = 0, \;\; f_y(a, b, c) = 0, \;\; f_z(a, b, c) = 0.$$

  2. Find the Hessian matrix of f

$$\mathbf{H}f = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

  Let

$$D_1 = f_{xx}, \qquad D_2 = det\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}, \qquad D_3 = det\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}.$$

  - If $D_1(a, b, c) > 0$, $D_2(a, b, c) > 0$, and $D_3(a, b, c) > 0$, then f has a local minimum at (a, b, c).
  - If $D_1(a, b, c) < 0$, $D_2(a, b, c) > 0$, and $D_3(a, b, c) < 0$, then f has a local maximum at (a, b, c).
  - In any other case where $D_3(a, b, c) \neq 0$, f has a saddle point at (a, b, c).