Math 2374

Final review 1

This review sheet is not meant to be your only form of studying. Understanding all the homework problems and lecture material are essential for success in the course. This review sheet only contains the key ideas of these sections.

- **Parametric equation** of a line through point (x_0, y_0, z_0) in direction of $\langle a, b, c\rangle$:

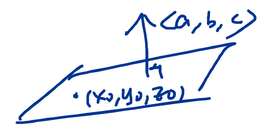
$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

direction

point

- **Equations of planes:** The equation of plane contains a point (x_0, y_0, z_0) and has a normal vector $\langle a, b, c\rangle$:

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$



or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

- Distance from a point (x_0, y_0, z_0) to a plane $ax + by + cz + d = 0$:

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

- Knowing graph of a function, Level sets, and Sections.

- Level sets: We call level curves for functions of two variables; and level surfaces for functions of three variables. Be able to sketch a few level sets as in the homework.

- **Linear approximation** of a function $f : \mathbb{R}^2 \to \mathbb{R}$ near a point (x_0, y_0).

(1st order Taylor appro.)

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0).$$

- Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (x_0, y_0). **Tangent plane** of the graphs of f at point $(x_0, y_0, f(x_0, y_0))$ is

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0).$$

→ normal vector $\langle f_x, f_y, -1\rangle$.

- Know that a curve can be parametrized by a function $c(t)$. Also, $c'(t)$ is the velocity of an object at position $c(t)$. Moreover, $c'(t)$ is **tangent vector** to the curve at time t.

- Be able to compute the tangent line to a curve.

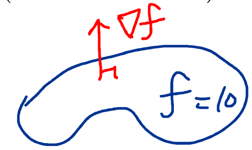
- (Chain Rule)

$$D(f \circ g)(x) = Df(g(x))Dg(x)$$

- The gradient ∇f:

  1. For scalar-valued function f, the gradient ∇f is like the matrix of partial derivatives Df, except that the gradient is a vector rather than a matrix.
  2. The gradient is a vector whose magnitude and direction have physical meaning:
     - The gradient points in the direction where f increase most rapidly.

3. For $f : \mathbb{R}^n \to \mathbb{R}$. The gradient ∇f is perpendicular to level sets of f, we can use the gradient to find tangent planes to surfaces.
   For example, when $n = 3$, the tangent plane of the level surface $f(x, y, z) = k$ (k is a constant) at point (x_0, y_0, z_0) is
   $$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$



- The directional derivative:
  1. The directional derivative $D_v f$ gives the rate of change of f in direction v.
  2. $D_v f(x) = \nabla f(x) \cdot v$ where v is a **unit vector**.

- (**Newton's second law**) Let $c(t)$ be the distance function of a moving particle of mass m in $\mathbb{R}^3$. Then the velocity $v(t) = c'(t)$ and the acceleration $a(t) = v'(t) = c''(t)$. Newton's second law is
  $$(force)\ F = ma.$$

- (**Arc length**) The (arc) length of the path $c(t) = (x(t), y(t), z(t))$ for $a \leq t \leq b$ is
  $$L = \int_a^b \|c'(t)\| dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt.$$

- Be able to compute bounds for iterated integrals, especially for the different orders of integration.

- The path integral of the scalar function f along the path $c(t), a \leq t \leq b$ is defined by
  $$\int_c f ds = \int_a^b f(c(t)) \|c'(t)\| dt.$$

- If $f(c)$ is density of wire, then $\int_c f ds$ is mass of wire.

- If $f = 1$, then $\int_c f ds = \int_c ds = \int_a^b \|c'(t)\| dt$ is length of c.

- The **line integral** of F along the path $c(t), a \leq t \leq b$ is defined by
  $$\int_c F \cdot d\mathbf{s} = \int_a^b F(c(t)) \cdot c'(t) dt.$$

- If F is a force field, then $\int_c F \cdot d\mathbf{s}$ is the **work done by the force field** on a particle moving along the path c.

- Let $c : [a, b] \to \mathbb{R}^3$ be the path. Then $\int_c \nabla f \cdot d\mathbf{s} = f(c(b)) - f(c(a))$.

Example 1. *Let* $f(x,y) = e^x(x^2+y^2)^{1/2}$.

1. *Compute an equation for the plane tangent to the graph* $z = f(x,y)$ *at* $x = 0, y = 1$, $z = f(0,1)$.

$$f(0,1) = e^0(1)^{1/2} = 1.$$

$$f_x(0,1) = e^0 \cdot 1 + 0 = 1, \quad f_x = e^x(x^2+y^2)^{1/2} + e^x \tfrac{1}{2}(x^2+y^2)^{-\frac{1}{2}} 2x$$

$$f_y(0,1) = 1. \quad , \quad f_y = e^x \tfrac{1}{2}(x^2+y^2)^{-\frac{1}{2}} 2y.$$

$$z = f(0,1) + f_x(0,1)(x-0) + f_y(0,1)(y-1).$$

$$= 1 + x + (y-1)$$

$$\underline{z = x+y.}$$

[1st order Taylor appro.].

2. *Find a linear approximation to* f *at the point* $(x,y) = (0,1)$. *Use it to estimate* $f(0.1, 1.1)$

$$T_1(x,y) = 1 + x + (y-1)$$

$$f(0.1, 1.1) \sim T_1(0.1, 1.1)$$

$$= 1 + 0.1 + (1.1 - 1)$$

$$= \underline{1.2}. \#$$

HW: Compute 2nd order Taylor approx., also use it to approximate $f(0.1, 1.1)$.

Example 2. *Suppose that atmospheric pressure at position (x, y, z) is given by the function*

$$A(x, y, z) = 100 - x^2 - e^{-xy}z^2.$$

1. *If you are currently at position* ~~(1, 2, 0)~~ (0, 2, 1)*, find the direction that you would need to move in order to decrease the atmospheric pressure as soon as possible. Write the answer in the form of a unit vector.*

$$\nabla A = \langle -2x + y e^{-xy} z^2, \; x e^{-xy} z^2, \; -2z e^{-xy} \rangle.$$

$$\nabla A(0,2,1) = \langle 2, 0, -2 \rangle$$

$$-\nabla A(0,2,1) = \langle -2, 0, 2 \rangle$$

unit vector : $\dfrac{-\nabla A(0,2,1)}{\|\nabla A(0,2,1)\|} = \dfrac{\langle -2, 0, 2 \rangle}{2\sqrt{2}} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right).$

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2. *At the point $(x, y, z) = (1, 2, 0)$, what is the directional derivative of A in the direction of the vector you found in (a).*

Directional derivatives : $v = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$$D_v A(1,2,0) = \nabla A(1,2,0) \cdot v$$

$$= (-2, 0, 0) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$

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