INCREASING STABILITY FOR THE DIFFUSION EQUATION

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ABSTRACT. We study the phenomenon of increasing stability of the diffusion and absorption coefficients in the diffuse equation. We derive some bounds which can be viewed as an evidence of increasing stability when the frequency is growing. These bounds hold under a-priori assumptions on the diffusion and absorption coefficients.

1. INTRODUCTION

In this paper, we are interested in studying the increase stability behavior in the diffuse equation when the frequency is growing. This problem arises in the diffusion approximation to optical tomography. Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a bounded domain with smooth boundary. We assume that the propagation of light through this medium can be modeled by the diffusion approximation

$$-\nabla \cdot \gamma \nabla u(x) + (D + ik)u(x) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where u describes the photon density in the medium, γ is the diffusion coefficient and D the absorption coefficient. For the detailed description of this model we refer the reader to the paper [2]. A result by Arridge and Lionheart [3] demonstrates that this problem is in general not uniquely solvable. However under suitable assumptions on the diffusion and absorption coefficients, the unique determination is possible, see [11] and reference there for details. This is also an exponentially unstable inverse problem. It has been observed experimentally in [8] that the resolution improves with increased frequency. The aim of this paper is to give the precise quantitative estimates of this behavior.

A short paper [5] published by A. P. Calderón in 1980 motivated many developments in inverse problems, in particular in the construction of CGO solutions of partial differential equations to solve inverse problems. The problem that Calderón considered was whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This inverse method is known as *Electrical Impedance Tomography* (EIT). EIT arises not only in geophysical prospections (See [26]) but also in medical imaging (See [12], [13] and [18]). We now describe more precisely the mathematical settings. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$ and the equation for the potential is given by

$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in } \Omega.$$

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Given $f \in H^{1/2}(\partial\Omega)$ on the boundary, the potential $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\begin{cases} \nabla \cdot \gamma \nabla u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The Dirichlet-to-Neumann map, or voltage-to-current map, is given by

$$\Lambda_\gamma f = \gamma \partial_\nu u|_{\partial\Omega},$$

where $\partial_\nu u = \nu \cdot \nabla u$ and ν is the unit outer normal to $\partial\Omega$. The well-known inverse problem is to recover the conductivity γ from the boundary measurement Λ_γ.

The uniqueness issue for C^2 conductivities was first settled by Sylvester and Uhlmann [24]. Later, the regularity of conductivity was relaxed to $3/2$ derivatives in some sense in [4] and [21]. Uniqueness for conductivities with conormal singularities in $C^{1,\varepsilon}$ was shown in [7]. See [25] for the detailed development. Recently, Haberman and Tataru [9] extended the uniqueness result to C^1 conductivities or small in the $W^{1,\infty}$ norm.

For the conductivity equation, a logarithmic stability estimate for conductivities was first obtained by Alessandrini [1]. Later Mandache [19] showed that this estimate is optimal. However, the logarithmic stability makes it difficult to design reliable reconstruction algorithms in practice since small errors in the data of the inverse problem result in large error in numerical reconstruction of physical properties of the medium. It has been observed numerically that the stability increases if one increases the frequency in some cases. These papers ([22], [23], [14], [15], [16], [20]) rigorously demonstrated the increasing stability phenomena in different settings. We briefly introduce two different methods of deriving bounds on the unknown coefficient. First, in [16], Isakov proved that the stability of the $W^{1,\infty}(\Omega)$ potential coefficient increases in the Schrödinger equation from boundary measurement when the frequency is growing. The idea is to use complex and real-valued geometrical optics solutions to deal with the low frequency and high frequency ranges, respectively. Second, Nagayasu, Uhlmann and Wang [20] considered the problem of stability for the refractive index in the space $H^s(\Omega), s > n/2+1$ in the acoustic equation. The stability estimates in [20] were valid for all the range of the frequency by using the complex geometrical optics (CGO) solutions constructed in [24]. Recently, Isakov, Nagayasu, Uhlmann and Wang [17] proved the increasing stability behavior in the Schrödinger equation by using similar computations in [20] and the CGO solutions in [10].

The novelty of Theorem 2.1, with respect to [16] and [20], is that we consider the stability estimate for the diffusion equation under less regular assumption on γ, that is, $\gamma \in C^{1,\varepsilon}(\overline{\Omega}), 0 < \varepsilon < 1$. For the conductivity equation, Haberman and Tataru [9] established the CGO solutions in the Bourgain-type spaces with C^1 conductivity. We adopt their methods to construct the CGO solutions for the diffusion equation. Having establishing the CGO solutions, we then substitute them into Lemma 4.1. The main difficulties to derive the bounds for the difference of the parameters are that the remainder terms of the CGO solutions with less regular γ only decay in average and the Bourgain-type spaces do not have Banach algebra property. Therefore, the method in [20] can not be applied directly to give the stability estimate in our case. To overcome these difficulties, we consider the low and high frequencies ranges of k separately which are dependent on the logarithm of the difference of the DN maps.

The outline of the paper is as follows. In section 2, we start with a more detailed description of the considered problem and state the main result. In section 3, we construct the complex geometrical optics solutions by following the idea of Haberman and Tataru [9]. Then we deduce a useful boundary integral estimate in section 4. The detailed proof of the main theorem is presented in section 5.

2. Main results

We assume that the diffusion coefficient $\gamma \in C^{1,\varepsilon}(\overline{\Omega}), 0 < \varepsilon < 1$. We consider the equation

$$-\nabla \cdot \gamma \nabla u(x) + (D + ik)u(x) = 0 \quad \text{in } \Omega \tag{2.1}$$

with the Dirichlet boundary data

$$u = g \quad \text{on } \partial\Omega. \tag{2.2}$$

For real-valued γ, the Dirichlet problem (2.1), (2.2) might fail to exist and be unique, so that the Dirichlet-to-Neumann map is not well-defined. Then one can consider replace this map by the Cauchy data with naturally defined norm (see [20]). We will assume that k is not a Dirichlet eigenvalue. Then Λ_γ is a continuous linear operator from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$. We denote its operator norm by $\|\Lambda_\gamma\|_*$.

Let $v = \sqrt{\gamma}u$. The equation (2.1) can be transformed into this Helmholtz type equation

$$-\Delta v(x) + Q(x)v(x) = 0 \quad \text{in } \Omega, \tag{2.3}$$

where Q can be formally defined by $Q = q + (D + ik)\gamma^{-1}$ with $q = \Delta\sqrt{\gamma}/\sqrt{\gamma}$.

Now we have the main result.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a bounded domain with smooth boundary and $2s > n + 3$. Let M and ε be real constants such that $M > 1$ and $0 < \varepsilon < 1$. Suppose that $\|\gamma_j\|_{C^{1,\varepsilon}(\overline{\Omega})} \leq M$ with $\gamma_j(x) > 1/M$ for all $x \in \Omega$ and $\operatorname{supp}(\gamma_1 - \gamma_2) \subset \Omega$. Let $\alpha > 2$ and*

$$E = -\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*$$

with $E \geq c$ for some constant $c > 0$. There exist constants $C > 0$ depending on $n, s, \varepsilon, \Omega, M$ and C_1 depending on n, Ω, M, such that if $k \geq 1$, then one has the following stability estimates: if $k^\alpha \leq C_1 E$, then

$$\begin{aligned} &\| \left(q_1 + (D_1 + ik)\gamma_1^{-1}\right) - \left(q_2 + (D_2 + ik)\gamma_2^{-1}\right) \|_{H^{-s}(\Omega)} \\ &\leq C(E + k^\alpha)^{\frac{-\varepsilon}{1+\varepsilon}} + Ck^2(E + k^\alpha)^{-1} + Ck(E + k^\alpha)^{(1-s)/\alpha} + CE^2\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{1/2}. \end{aligned} \tag{2.4}$$

On the other hand, if $k^\alpha \geq C_1 E$, then

$$\begin{aligned} &\| \left(q_1 + (D_1 + ik)\gamma_1^{-1}\right) - \left(q_2 + (D_2 + ik)\gamma_2^{-1}\right) \|_{H^{-s}(\Omega)} \\ &\leq C(E + k^\alpha)^{\frac{-\varepsilon}{1+\varepsilon}} + Ck^2(E + k^\alpha)^{-1} + Ck(E + k^\alpha)^{(1-s)/\alpha} + Ck^{2\alpha}e^{Ck^\alpha}\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*. \end{aligned} \tag{2.5}$$

Remark 2.1. The bound (2.4) is the stability estimate for the associated difference in the absorption and diffusion parameters. In other words, the bound vanishes when $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*$ goes to zero. Moreover, the estimate (2.4) consists of two parts-Hölder and the logarithmic estimates. When k is growing, the logarithmic component decreases and the Hölder part becomes dominant. This bound can be viewed as an evidence of increasing stability in recovering absorption and diffusion coefficients for larger frequencies k.

On the other hand, for high frequency range of k, one can derive the estimate (2.5) which consists of two parts-Lipschitz and the logarithmic estimates. The logarithmic part also goes to zero as k grows.

3. Complex geometrical optics solutions

In this section, we adopt Bourgain-type spaces introduced by Haberman and Tataru in [9] to construct CGO solutions to (2.1) with less regular diffusion coefficient.

Let $\zeta \in \mathbb{C}^n, n \geq 3$ and let p_ζ denote the polynomial

$$p_\zeta(\xi) = -|\xi|^2 + 2i\zeta \cdot \xi,$$

which is the symbol of the operator $\Delta_\zeta = \Delta + 2\zeta \cdot \nabla$. For any $b \in \mathbb{R}$, we define spaces $\dot{X}_\zeta^b$ and X_ζ^b by the norm

$$\|u\|_{\dot{X}_\zeta^b} = \||p_\zeta(\xi)|^b \hat{u}(\xi)\|_{L^2}$$

and

$$\|u\|_{X_\zeta^b} = \|(|\zeta| + |p_\zeta(\xi)|)^b \hat{u}(\xi)\|_{L^2},$$

respectively. We will only use the cases where $b \in \{1/2, -1/2\}$. Note that $\dot{X}_\zeta^{-1/2}$ can be identified as the dual space of $\dot{X}_\zeta^{1/2}$. One feature of these spaces is that the operator Δ_ζ^{-1} is a bounded linear operator from $\dot{X}_\zeta^{-1/2}$ to $\dot{X}_\zeta^{1/2}$ with norm

$$\|\Delta_\zeta^{-1}\|_{\mathcal{L}\left(\dot{X}_\zeta^{-1/2} \to \dot{X}_\zeta^{1/2}\right)} = 1.$$

Assume that the conductivity γ_j is in the space $C^{1,\varepsilon}(\overline{\Omega}), 0 < \varepsilon < 1$ and satisfies

$$\operatorname{supp}(\gamma_1 - \gamma_2) \subset \Omega$$

with $\|\gamma_j\|_{C^{1,\varepsilon}(\overline{\Omega})} \leq M$ and $\gamma_j(x) > 1/M$. By Lemma 2.3 in [6], there exist a ball B with center 0 and radius R and σ in $C^{1,\varepsilon}(\mathbb{R}^n)$ such that $\overline{\Omega} \subset B$, $\gamma_j = \sigma_j|_{\overline{\Omega}}$ and

$$\operatorname{supp}(\sigma_j - 1) \subset B$$

for $j = 1, 2$. Now we use the same notation to express that $\gamma \in C^{1,\varepsilon}(\mathbb{R}^n)$. Moreover, we assume that $\|D\|_{L^\infty(\Omega)} \leq M$ and extend the coefficient $D + ik$ by zero to $\mathbb{R}^n \setminus \Omega$. Hence the equation (2.3) can be extended to $\mathbb{R}^n$ in the following sense:

$$-\Delta v + Q(x)v(x) = 0 \quad \text{in } \mathbb{R}^n, \tag{3.1}$$

where $Q = q + (D + ik)\gamma^{-1}$ with $q = \Delta\sqrt{\gamma}/\sqrt{\gamma}$. Suppose that the CGO solutions of (3.1) has the form

$$v(x) = e^{\zeta \cdot x}(1 + \psi(x)) \quad \text{in } \mathbb{R}^n \tag{3.2}$$

with $\zeta \in \mathbb{C}^n$ satisfies $\zeta \cdot \zeta = 0$. We define Q as a linear functional and m_Q as an operator by

$$\langle Q|\phi_1\rangle := -\int_{\mathbb{R}^n} \nabla\gamma^{1/2} \cdot \nabla\left(\gamma^{-1/2}\phi_1\right) dx + \int_\Omega (D + ik)\gamma^{-1}\phi_1 dx$$

and

$$\langle m_Q\phi_1|\phi_2\rangle := -\int_{\mathbb{R}^n} \nabla\gamma^{1/2} \cdot \nabla\left(\gamma^{-1/2}\phi_1\phi_2\right) dx + \int_\Omega (D + ik)\gamma^{-1}\phi_1\phi_2 dx$$

for any $\phi_1, \phi_2 \in \dot{X}_\zeta^{1/2}$. By substituting (3.2) into (3.1), the remainder ψ needs to satisfy

$$-\Delta_\zeta\psi + m_Q\psi = -Q \quad \text{in } \mathbb{R}^n$$

with $\zeta\cdot\zeta=0$. We define ϕ_B to be a fixed Schwartz function and write $u_B=\phi_B u$. The following estimates in Lemma 3.1 and 3.2 will be used throughout the paper. The proof of these estimates can be found in [9].

Lemma 3.1. *There exists a constant C which is independent of ζ such that one has*

$$\begin{aligned}
&\|u_B\|_{L^2(\mathbb{R}^n)}\le C|\zeta|^{-1/2}\|u\|_{\dot X^{1/2}_\zeta}, && \|u_B\|_{H^{1/2}(\mathbb{R}^n)}\le C\|u\|_{\dot X^{1/2}_\zeta};\\
&\|u_B\|_{H^1(\mathbb{R}^n)}\le C|\zeta|^{1/2}\|u\|_{\dot X^{1/2}_\zeta}, && \|u\|_{X^{-1/2}_\zeta}\le C|\zeta|^{-1/2}\|u\|_{L^2(\mathbb{R}^n)};\\
&\|u_B\|_{\dot X^{-1/2}_\zeta}\le C\|u\|_{X^{-1/2}_\zeta}, && \|u_B\|_{X^{1/2}_\zeta}\le C\|u\|_{\dot X^{1/2}_\zeta}.
\end{aligned}$$

Lemma 3.2. *If f is a bounded function and $\zeta_j\in\mathbb{C}^n$ are such that $\zeta_j\cdot\zeta_j=0$ and $|\zeta_1|=|\zeta_2|$, then*

$$\left|\int f u_B v_B dx\right|\le C\tau^{-1}\|f\|_{L^\infty}\|u\|_{\dot X^{1/2}_{\zeta_1}}\|v\|_{\dot X^{1/2}_{\zeta_2}}, \tag{3.3}$$

where $|\zeta_1|=|\zeta_2|=2^{1/2}\tau$ and the constant C is independent of τ.

Now we are ready to prove the existence of the remainder ψ.

Theorem 3.3. *Suppose that $k\ge 1$. Let $\zeta\in\mathbb{C}^n$ satisfy $\zeta\cdot\zeta=0$. Then there exists a positive constant C_* depending only on n,Ω and M such that if*

$$|\zeta|\ge C_* k,$$

then there exists a unique solution $\psi\in\dot X^{1/2}_\zeta$ to the equation

$$-\Delta_\zeta\psi+m_Q\psi=-Q \tag{3.4}$$

satisfying the estimate

$$\|\psi\|_{\dot X^{1/2}_\zeta}\le C\|Q\|_{\dot X^{-1/2}_\zeta},$$

where C is independent of k and $|\zeta|$.

Proof. By using the Neumann series argument (see [24]), we can show the existence of $\psi\in\dot X^{1/2}_\zeta$ which satisfies

$$\|\psi\|_{\dot X^{1/2}_\zeta}\le\|(I-\Delta_\zeta^{-1}(m_Q))^{-1}\|_{\mathcal{L}\left(\dot X^{1/2}_\zeta\to\dot X^{1/2}_\zeta\right)}\|\Delta_\zeta^{-1}(Q)\|_{\dot X^{1/2}_\zeta}$$

for $|\zeta|$ large enough and

$$\|m_Q\|_{\mathcal{L}\left(\dot X^{1/2}_\zeta\to\dot X^{-1/2}_\zeta\right)}<1. \tag{3.5}$$

To prove (3.5), the definition of the operator m_Q gives

$$\begin{aligned}
\langle m_Q\phi_1|\phi_2\rangle &= -\int_{\mathbb{R}^n}\nabla\gamma^{1/2}\cdot\nabla\left(\gamma^{-1/2}\phi_1\phi_2\right)dx+\int_\Omega(D+ik)\gamma^{-1}\phi_1\phi_2dx\\
&= -\int_{\mathbb{R}^n}\nabla\gamma^{1/2}\cdot\nabla\gamma^{-1/2}\phi_1\phi_2dx-\int_{\mathbb{R}^n}\gamma^{-1/2}\nabla\gamma^{1/2}\cdot\nabla(\phi_1\phi_2)dx\\
&\quad+\int_\Omega(D+ik)\gamma^{-1}\phi_1\phi_2dx\\
&=: I_1+I_2+I_3
\end{aligned}$$

for any $\phi_1, \phi_2 \in \dot{X}_\zeta^{1/2}$. We first consider the terms I_1 and I_3. Applying (3.3), one has

$$|I_1| \le \frac{C}{|\zeta|} \|\nabla\gamma^{1/2} \cdot \nabla\gamma^{-1/2}\|_{L^\infty(B)} \|\phi_1\|_{\dot{X}_\zeta^{1/2}} \|\phi_2\|_{\dot{X}_\zeta^{1/2}} \le \frac{C}{|\zeta|} \|\phi_1\|_{\dot{X}_\zeta^{1/2}} \|\phi_2\|_{\dot{X}_\zeta^{1/2}}$$

and

$$|I_3| \le \frac{C}{|\zeta|} \|(D+ik)\gamma^{-1}\|_{L^\infty(\Omega)} \|\phi_1\|_{\dot{X}_\zeta^{1/2}} \|\phi_2\|_{\dot{X}_\zeta^{1/2}} \le \frac{Ck}{|\zeta|} \|\phi_1\|_{\dot{X}_\zeta^{1/2}} \|\phi_2\|_{\dot{X}_\zeta^{1/2}}.$$

Here C is independent of k and $|\zeta|$.

Let Ψ be a smooth function in $\mathbb{R}^n$ with support in the unit ball and $\int \Psi dx = 1$. Define $\Psi_h = h^{-n}\Psi(x/h)$ with $h > 0$. We have

$$\begin{aligned} |I_2| \le & \left| \int_{\mathbb{R}^n} \Psi_h * \left(\gamma^{-1/2}\nabla\gamma^{1/2}\right) \cdot \nabla(\phi_1\phi_2) dx \right| \\ & + \left| \int_{\mathbb{R}^n} \left(\Psi_h * (\gamma^{-1/2}\nabla\gamma^{1/2}) - \gamma^{-1/2}\nabla\gamma^{1/2}\right) \cdot \nabla(\phi_1\phi_2) dx \right|. \end{aligned}$$

By Lemma 2.3 in [9], we get

$$\left| \int_{\mathbb{R}^n} \Psi_h * \left(\gamma^{-1/2}\nabla\gamma^{1/2}\right) \cdot \nabla(\phi_1\phi_2) dx \right| \le C|\zeta|^{-1} h^{-1} \|\phi_1\|_{\dot{X}_\zeta^{1/2}} \|\phi_2\|_{\dot{X}_\zeta^{1/2}}$$

and

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \left(\Psi_h * (\gamma^{-1/2}\nabla\gamma^{1/2}) - \gamma^{-1/2}\nabla\gamma^{1/2}\right) \cdot \nabla(\phi_1\phi_2) dx \right| \\ & \le \|\Psi_h * (\gamma^{-1/2}\nabla\gamma^{1/2}) - \gamma^{-1/2}\nabla\gamma^{1/2}\|_{L^\infty(\mathbb{R}^n)} \|\phi_1\|_{\dot{X}_\zeta^{1/2}} \|\phi_2\|_{\dot{X}_\zeta^{1/2}} \\ & \le h^\varepsilon \|\phi_1\|_{\dot{X}_\zeta^{1/2}} \|\phi_2\|_{\dot{X}_\zeta^{1/2}}, \end{aligned}$$

where the last inequality is based on the fact that $\gamma \in C^{1+\varepsilon}(\mathbb{R}^n)$. Let $h = |\zeta|^{-1/(1+\varepsilon)}$, then

$$\|m_Q\|_{\mathcal{L}\left(\dot{X}_\zeta^{1/2} \to \dot{X}_\zeta^{-1/2}\right)} \le C\left(|\zeta|^{-1} + |\zeta|^{-\varepsilon/(1+\varepsilon)} + k|\zeta|^{-1}\right), \tag{3.6}$$

where C depends on n, Ω and M. Taking $|\zeta| \ge C_* k$ with $C_* > 0$ sufficiently large, the estimate (3.5) holds. Since

$$\|\Delta_\zeta^{-1}(m_Q)\|_{\mathcal{L}\left(\dot{X}_\zeta^{1/2} \to \dot{X}_\zeta^{1/2}\right)} \le \|m_Q\|_{\mathcal{L}\left(\dot{X}_\zeta^{1/2} \to \dot{X}_\zeta^{-1/2}\right)} < 1$$

implies the operator $(I - \Delta_\zeta^{-1}(m_Q))^{-1}$ exists, we conclude the solution of (3.4) exists and is unique. The proof is complete. □

Now we address the behavior of ψ as $|\zeta| \to \infty$. From the theorem above, it is sufficient to observe that if $\|Q\|_{\dot{X}_\zeta^{-1/2}} \to 0$ when $|\zeta| \to \infty$.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi = 1$ on B. By the definition of the functional Q, for $\phi \in \dot{X}_\zeta^{1/2}$, we have

$$
\begin{aligned}
|\langle Q|\phi\rangle| &= \left| -\int_{\mathbb{R}^n} \nabla\gamma^{1/2} \cdot \nabla\left(\gamma^{-1/2}\phi\right) dx + \int_\Omega (D+ik)\gamma^{-1}\phi dx \right| \\
&\le \frac{1}{4}\int_{\mathbb{R}^n} |\nabla \log\gamma|^2 \phi dx + \frac{1}{2}\left|\int_{\mathbb{R}^n} \nabla\log\gamma\cdot\nabla\phi dx\right| + \left|\int_\Omega (D+ik)\gamma^{-1}\phi dx\right| \\
&\le C\left(|\zeta|^{-1/2}\|\nabla\log\gamma\|^2_{L^\infty(\mathbb{R}^n)} + \|\varphi\Delta\log\gamma\|_{\dot{X}_\zeta^{-1/2}} + k|\zeta|^{-1/2}\right)\|\phi\|_{\dot{X}_\zeta^{1/2}}.
\end{aligned}
$$

Note that the last inequality is obtained by using Hölder inequality and $\|\varphi\phi\|_{L^2(\mathbb{R}^n)} \le C|\zeta|^{-1/2}\|\phi\|_{\dot{X}_\zeta^{1/2}}$. Thus,

$$
\|Q\|_{\dot{X}_\zeta^{-1/2}} \le C\left(|\zeta|^{-1/2} + \|\varphi\Delta\log\gamma\|_{\dot{X}_\zeta^{-1/2}} + k|\zeta|^{-1/2}\right) \tag{3.7}
$$

with C depends on n, Ω and M. Applying the fact that $|\xi|^2/2 \le |p_\zeta(\xi)| \le 3|\xi|^2/2$ when $4|\zeta| \le |\xi|$ and by direct computation, one has

$$
\begin{aligned}
\|\varphi\Delta\log\gamma\|_{\dot{X}_\zeta^{-1/2}} &\le C\|\Delta\log\gamma\|_{X_\zeta^{-1/2}} \\
&= C\left(\int_{|\xi|<4|\zeta|} (|\zeta| + |p_\zeta(\xi)|)^{-1}|\xi|^2|\widehat{\nabla\log}\gamma(\xi)|^2 dx\right)^{1/2} \\
&\quad + C\left(\int_{|\xi|\ge 4|\zeta|} (|\zeta| + |p_\zeta(\xi)|)^{-1}|\xi|^2|\widehat{\nabla\log}\gamma(\xi)|^2 dx\right)^{1/2} \\
&= C\left(\int_{|\xi|<4|\zeta|} |\zeta||\widehat{\nabla\log}\gamma(\xi)|^2 dx\right)^{1/2} + C\left(\int_{|\xi|\ge 4|\zeta|} |\widehat{\nabla\log}\gamma(\xi)|^2 dx\right)^{1/2} \\
&= C(1 + |\zeta|^{1/2})\|\nabla\log\gamma\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

Thus, we have

$$
\|Q\|_{\dot{X}_\zeta^{-1/2}} \le C\left(|\zeta|^{-1/2} + |\zeta|^{1/2}) + k|\zeta|^{-1/2}\right). \tag{3.8}
$$

Since $\|Q\|_{\dot{X}_\zeta^{-1/2}}$ does not decay as $|\zeta|$ grows, Haberman and Tataru studied this estimate in average.

Let $r \ge 0$ and $\eta \in S^{n-1}$. We set

$$
\zeta = \tau\eta_1 + i(\beta - r\eta/2),
$$

where $\tau > 0$, $\beta \in \mathbb{R}^n$ and $\eta_1 \in S^{n-1}$ satisfy

$$
\eta_1 \cdot \beta = \eta_1 \cdot \eta = \eta \cdot \beta = 0, \quad \tau^2 = |\beta|^2 + r^2/4.
$$

The vector ζ is chosen so that $\zeta \cdot \zeta = 0$ and $|\zeta|^2 = 2\tau^2$. Then we have the following lemma.

Lemma 3.4. *Suppose that $k \ge 1$. Then there exists a constant C depending on n, Ω, M such that*

$$
\frac{1}{\lambda}\int_{S^{n-1}}\int_\lambda^{2\lambda} \|Q\|^2_{\dot{X}_\zeta^{-1/2}} d\tau d\eta_1 \le C\left(\left(1 + \langle r\rangle^2\lambda^{-1}\right)\lambda^{-\varepsilon/(1+\varepsilon)} + k^2\lambda^{-1}\right)
$$

when λ is sufficiently large.

Proof. From (3.7), we have known that

$$\|Q\|_{\dot{X}_\zeta^{-1/2}} \leq C\left(|\zeta|^{-1/2} + \|\varphi\Delta\log\gamma\|_{\dot{X}_\zeta^{-1/2}} + k|\zeta|^{-1/2}\right)$$

with C depends on n, Ω and M. To show the second term of (3.7) decays as $|\zeta|$ increase, we consider the average

$$\begin{aligned}
&\frac{1}{\lambda}\int_{S^{n-1}}\int_\lambda^{2\lambda} \|\varphi\Delta\log\gamma\|^2_{\dot{X}_\zeta^{-1/2}} d\tau d\eta_1 \\
&\leq C\frac{1}{\lambda}\int_{S^{n-1}}\int_\lambda^{2\lambda} \|\varphi\nabla\cdot(\Psi_h * \nabla\log\gamma)\|^2_{\dot{X}_\zeta^{-1/2}} d\tau d\eta_1 \\
&\quad + C\frac{1}{\lambda}\int_{S^{n-1}}\int_\lambda^{2\lambda} \|\varphi\nabla\cdot(\Psi_h * \nabla\log\gamma - \nabla\log\gamma)\|^2_{\dot{X}_\zeta^{-1/2}} d\tau d\eta_1.
\end{aligned}$$

We deduce

$$\begin{aligned}
&\frac{1}{\lambda}\int_{S^{n-1}}\int_\lambda^{2\lambda} \|\varphi\nabla\cdot(\Psi_h * \nabla\log\gamma)\|^2_{\dot{X}_\zeta^{-1/2}} d\tau d\eta_1 \\
&\leq C\frac{1}{\lambda}\|\nabla\cdot(\Psi_h * \nabla\log\gamma)\|^2_{L^2(\mathbb{R}^n)} \leq C\frac{1}{h^2\lambda}\|\nabla\log\gamma\|^2_{L^\infty(\mathbb{R}^n)}
\end{aligned}$$

and

$$\begin{aligned}
&C\frac{1}{\lambda}\int_{S^{n-1}}\int_\lambda^{2\lambda} \|\varphi\nabla\cdot(\Psi_h * \nabla\log\gamma - \nabla\log\gamma)\|^2_{\dot{X}_\zeta^{-1/2}} d\tau d\eta_1 \\
&\leq C\left(1 + \langle r\rangle^2\lambda^{-1}\right)\|\Psi_h * \nabla\log\gamma - \nabla\log\gamma\|^2_{L^2(\mathbb{R}^n)} \leq C\left(1 + \langle r\rangle^2\lambda^{-1}\right)h^{2\varepsilon}.
\end{aligned}$$

from Lemma 3.1 in [9]. Note that $\langle r\rangle = (1 + r^2)^{1/2}$. Summing up, we have

$$\frac{1}{\lambda}\int_{S^{n-1}}\int_\lambda^{2\lambda} \|Q\|^2_{\dot{X}_\zeta^{-1/2}} d\tau d\eta_1 \leq C\left(\lambda^{-1} + h^{-2}\lambda^{-1} + \left(1 + \langle r\rangle^2\lambda^{-1}\right)h^{2\varepsilon} + k^2\lambda^{-1}\right).$$

Taking $h = \lambda^{-1/(2+2\varepsilon)}$, since $0 < \varepsilon < 1$ and $\lambda \geq 1$, we complete the proof. □

4. A boundary integral estimate

In this section we derive a useful boundary integral estimate.

Lemma 4.1. *Let $\gamma_j \in C^{1,\varepsilon}(\overline{\Omega})$ be two given functions with positive lower bound and* $\mathrm{supp}(\gamma_1 - \gamma_2) \subset \Omega$ *for $j = 1, 2$. Then for any $u_j \in H^1(\Omega)$ weak solution of $-\nabla\cdot(\gamma_j\nabla u_j) + (D_j + ik)u_j = 0$ in Ω, one has*

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} \nabla\gamma_2^{1/2}\cdot\nabla\left(\gamma_2^{-1/2}v_1v_2\right)dx - \int_{\mathbb{R}^n}\nabla\gamma_1^{1/2}\cdot\nabla\left(\gamma_1^{-1/2}v_1v_2\right)dx \\
&+ \int_\Omega\left((D_1 + ik)\gamma_1^{-1} - (D_2 + ik)\gamma_2^{-1}\right)v_1v_2dx\Big| \\
&\leq \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*\|u_1\|_{H^1(\Omega)}\|u_2\|_{H^1(\Omega)}
\end{aligned} \tag{4.1}$$

with $v_j = \gamma_j^{1/2}u_j$.

Proof. Since $\gamma_1 = \gamma_2$ on $\partial\Omega$, applying integration by parts twice, we obtain that

$$\begin{aligned}
\langle \Lambda_{\gamma_1} u_1, u_2 \rangle &= \int_\Omega \gamma_1 \nabla u_1 \cdot \nabla u_2 dx + \int_\Omega (D_1 + ik) u_1 u_2 dx \\
&= \int_\Omega \gamma_1 \nabla u_1 \cdot \nabla \left(\gamma_1^{-1/2} v_2 \right) dx + \int_\Omega \gamma_1 \nabla u_1 \cdot \nabla \left(\left(\gamma_2^{-1/2} - \gamma_1^{-1/2} \right) v_2 \right) dx \\
&\quad + \int_\Omega (D_1 + ik) u_1 u_2 dx \\
&= \int_\Omega \gamma_1 \nabla u_1 \cdot \nabla \left(\gamma_1^{-1/2} v_2 \right) dx - \int_\Omega (D_1 + ik) u_1 \left(\gamma_2^{-1/2} - \gamma_1^{-1/2} \right) v_2 dx \\
&\quad + \int_\Omega (D_1 + ik) u_1 u_2 dx, \\
&= \int_\Omega \gamma_1 \nabla u_1 \cdot \nabla \left(\gamma_1^{-1/2} v_2 \right) dx + \int_\Omega (D_1 + ik) \gamma_1^{-1} v_1 v_2 dx,
\end{aligned}$$

where $v_j = \gamma_j^{1/2} u_j$. Similarly, we have

$$\langle \Lambda_{\gamma_2} u_2, u_1 \rangle = \int_\Omega \gamma_2 \nabla u_2 \cdot \nabla \left(\gamma_2^{-1/2} v_1 \right) dx + \int_\Omega (D_2 + ik) \gamma_2^{-1} v_1 v_2 dx.$$

Since $\langle \Lambda_{\gamma_j} f, g \rangle = \langle \Lambda_{\gamma_j} g, f \rangle$ and

$$\int_\Omega \gamma_1 \nabla \left(\gamma_1^{-1/2} v_1 \right) \cdot \nabla \left(\gamma_1^{-1/2} v_2 \right) dx = \int_\Omega \nabla v_1 \cdot \nabla v_2 dx - \int_\Omega \nabla \gamma_1^{1/2} \cdot \nabla \left(\gamma_1^{-1/2} v_1 v_2 \right) dx,$$

it follows that

$$\begin{aligned}
&\langle (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) u_1, u_2 \rangle \\
&= \int_\Omega \nabla \gamma_2^{1/2} \cdot \nabla \left(\gamma_2^{-1/2} v_1 v_2 \right) dx - \int_\Omega \nabla \gamma_1^{1/2} \cdot \nabla \left(\gamma_1^{-1/2} v_1 v_2 \right) dx \\
&\quad + \int_\Omega \left((D_1 + ik) \gamma_1^{-1} - (D_2 + ik) \gamma_2^{-1} \right) v_1 v_2 dx,
\end{aligned}$$

which implies (4.1).

□

5. Proof of the main theorem

This section is devoted to the proof of Theorem 2.1. Let $r \geq 0$ and $\eta \in S^{n-1}$. We pick two vectors $\beta \in \mathbb{R}^n$, $\eta_1 \in S^{n-1}$ such that

$$\beta \cdot \eta_1 = \beta \cdot \eta = \eta \cdot \eta_1 = 0, \quad |\tau|^2 = |\beta|^2 + r^2/4.$$

Denote

$$\zeta_1 = \tau \eta_1 + i(\beta - r\eta/2), \quad \zeta_2 = -\tau \eta_1 - i(\beta + r\eta/2).$$

The two vectors ζ_l are chosen so that $\zeta_l \cdot \zeta_l = 0$, $\zeta_1 + \zeta_2 = -ir\eta$, $|\zeta_l|^2 = 2\tau^2$ for $l = 1, 2$. By Theorem 3.3, if

$$|\zeta_l| \geq C_* k$$

or

$$\tau \geq 2^{-1/2} C_* k$$

(C_* is the constant given in Theorem 3.3), we can construct CGO solutions v_l to the equation (3.1) with $Q = Q_l$ of the form

$$v_l(x) = e^{\zeta_l \cdot x}(1 + \psi_l(x))$$

with $\|\psi_l\|_{\dot{X}^{1/2}_{\zeta_l}} \le C\|Q_l\|_{\dot{X}^{-1/2}_{\zeta_l}}$ for $l = 1, 2$. Thus, $u_l(x) = \gamma_l^{-1/2} v_l(x) = \gamma_l^{-1/2} e^{\zeta_l \cdot x}(1+\psi_l(x))$.

Lemma 5.1. *Let u_l be the solutions to the equations $-\nabla \cdot (\gamma_l \nabla u_l) + (D_l + ik)u_l = 0$. Then one has*

$$\|u_l\|^2_{H^1(\Omega)} \le C|\zeta_l|^2 e^{2R|\zeta_l|}, \tag{5.1}$$

where $|\zeta_l| \ge \max\{1, C_* k\}$ and C depends on n, Ω and M.

Proof. Recall that B is a open ball with center at the origin and radius R such that $\overline{\Omega} \subset B$. We define the function $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\varphi = 1$ on B. From Lemma 3.1, we have $\|\psi_l\|^2_{L^2(\Omega)} \le C|\zeta_l|^{-1}\|\psi_l\|^2_{\dot{X}^{1/2}_{\zeta_l}}$, which implies that

$$\begin{aligned}
\|u_l\|^2_{L^2(\Omega)} &= \|\gamma_l^{-1/2} e^{\zeta_l \cdot x}(1 + \psi_l)\|^2_{L^2(\Omega)} \\
&\le Ce^{2R|\zeta_l|}\left(1 + \|\psi_l\|^2_{L^2(\Omega)}\right) \\
&\le Ce^{2R|\zeta_l|}\left(1 + |\zeta_l|^{-1}\|\psi_l\|^2_{\dot{X}^{1/2}_{\zeta_l}}\right).
\end{aligned}$$

Using the fact that $|\xi|^2/2 \le |p_\zeta(\xi)| \le 3|\xi|^2/2$ when $4|\zeta| \le |\xi|$, we can deduce

$$\begin{aligned}
\|\nabla\psi_l\|^2_{L^2(\Omega)} \le \|\nabla(\varphi\psi_l)\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\xi|^2 |\widehat{\varphi\psi_l}|^2 d\xi \\
&\le C\int_{|\xi|<4|\zeta_l|} |\zeta_l|^2 |\widehat{\varphi\psi_l}|^2 d\xi + C\int_{|\xi|\ge 4|\zeta_l|} |p_{\zeta_l}(\xi)||\widehat{\varphi\psi_l}|^2 d\xi \\
&\le C|\zeta_l|^2\|\varphi\psi_l\|^2_{L^2(\mathbb{R}^n)} + C\|\varphi\psi_l\|^2_{X^{1/2}_{\zeta_l}} \\
&\le C(|\zeta_l| + 1)\|\psi_l\|^2_{\dot{X}^{1/2}_{\zeta_l}}
\end{aligned}$$

from Lemma 3.1. Thus, we have

$$\begin{aligned}
\|\nabla u_l\|^2_{L^2(\Omega)} &\le \|\zeta_l e^{\zeta_l \cdot x}\gamma_l^{-1/2}(1 + \psi_l)\|^2_{L^2(\Omega)} + \|e^{\zeta_l \cdot x}\nabla\gamma_l^{-1/2}(1 + \psi_l)\|^2_{L^2(\Omega)} \\
&\quad + \|e^{\zeta_l \cdot x}\gamma_l^{-1/2}(\nabla\psi_l)\|^2_{L^2(\Omega)} \\
&\le Ce^{2R|\zeta_l|}\left(1 + |\zeta_l| + |\zeta_l|\|\psi_l\|^2_{\dot{X}^{1/2}_{\zeta_l}}\right).
\end{aligned}$$

From (3.8) and Theorem 3.3, we have that

$$\|\psi_l\|_{\dot{X}^{1/2}_{\zeta_l}} \le C\|Q_l\|_{\dot{X}^{-1/2}_{\zeta_l}} \le C(|\zeta_l|^{1/2} + k|\zeta_l|^{-1/2}),$$

then the estimate

$$\|u_l\|^2_{H^1(\Omega)} \le Ce^{2R|\zeta_l|}(1 + |\zeta_l|^{-1}(|\zeta_l| + k^2|\zeta_l|^{-1})) + Ce^{2R|\zeta_l|}(1 + |\zeta_l| + |\zeta_l|(|\zeta_l| + k^2|\zeta_l|^{-1}))$$

leads to (5.1).

□

We substitute CGO solutions v_l into the left hand side of (4.1), then we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla\gamma_2^{1/2}\cdot\nabla\left(\gamma_2^{-1/2}v_1v_2\right)dx - \int_{\mathbb{R}^n} \nabla\gamma_1^{1/2}\cdot\nabla\left(\gamma_1^{-1/2}v_1v_2\right)dx\\
&+\int_\Omega \left((D_1+ik)\gamma_1^{-1}-(D_2+ik)\gamma_2^{-1}\right)v_1v_2dx\\
&=\langle Q_2-Q_1|e^{-ir\eta\cdot x}\rangle+\langle Q_2-Q_1|e^{-ir\eta\cdot x}(\psi_1+\psi_2)\rangle\\
&\quad+\langle m_{Q_1}\psi_1|e^{-ir\eta\cdot x}\psi_2\rangle-\langle m_{Q_2}\psi_2|e^{-ir\eta\cdot x}\psi_1\rangle. \qquad (5.2)
\end{aligned}$$

Using the estimate (3.17) in [6], that is, $\|\varphi e^{-ir\eta\cdot x}\psi_1\|_{X_{\zeta_1}^{1/2}} \le C\langle r\rangle^{1/2}\|\psi_1\|_{\dot X_{\zeta_1}^{1/2}}$ and Lemma 3.1, we deduce that

$$\begin{aligned}
|\langle Q_j|e^{-ir\eta\cdot x}(\psi_1+\psi_2)\rangle| &= |\langle Q_j|\varphi e^{-ir\eta\cdot x}(\psi_1+\psi_2)\rangle|\\
&\le \|Q_j\|_{X_{\zeta_j}^{-1/2}}\left(\|\varphi e^{-ir\eta\cdot x}\psi_1\|_{X_{\zeta_1}^{1/2}}+\|\varphi e^{-ir\eta\cdot x}\psi_2\|_{X_{\zeta_2}^{1/2}}\right)\\
&\le C\langle r\rangle^{1/2}\|Q_j\|_{\dot X_{\zeta_j}^{-1/2}}\left(\|\psi_1\|_{\dot X_{\zeta_1}^{1/2}}+\|\psi_2\|_{\dot X_{\zeta_2}^{1/2}}\right). \qquad (5.3)
\end{aligned}$$

Since $\gamma_j\in C^{1,\varepsilon}(\mathbb{R}^n)$, by the operator norm (3.6) of m_Q, we obtain

$$|\langle m_{Q_1}\psi_1|e^{-ir\eta\cdot x}\psi_2\rangle| \le C\left(\tau^{-1}+\tau^{-\varepsilon/(1+\varepsilon)}+k\tau^{-1}\right)\langle r\rangle^{1/2}\|\psi_1\|_{\dot X_{\zeta_1}^{1/2}}\|\psi_2\|_{\dot X_{\zeta_2}^{1/2}} \qquad (5.4)$$

with $\tau>1$. Combining (4.1), (5.2), (5.3) and (5.4), it follows that

$$\begin{aligned}
|\langle Q_2-Q_1|e^{-ir\eta\cdot x}\rangle| &\le C\langle r\rangle^{1/2}\sum_{l,j=1}^{2}\|Q_j\|_{\dot X_{\zeta_j}^{-1/2}}\|\psi_l\|_{\dot X_{\zeta_l}^{1/2}}\\
&\quad+C\left(\tau^{-1}+\tau^{-\varepsilon/(1+\varepsilon)}+k\tau^{-1}\right)\langle r\rangle^{1/2}\|\psi_1\|_{\dot X_{\zeta_1}^{1/2}}\|\psi_2\|_{\dot X_{\zeta_2}^{1/2}}\\
&\quad+\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_*\|u_1\|_{H^1(\Omega)}\|u_2\|_{H^1(\Omega)}. \qquad (5.5)
\end{aligned}$$

Denote $A=\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_*^2$. Since $\|u_l\|_{H^1(\Omega)}^2\le C|\zeta_l|^2e^{2R|\zeta_l|}$, $|\zeta_l|^2=2\tau^2$ and $\|\psi_l\|_{\dot X_{\zeta_l}^{1/2}}\le C\|Q_l\|_{\dot X_{\zeta_l}^{-1/2}}$, we have

$$\begin{aligned}
|\langle Q_2-Q_1|e^{-ir\eta\cdot x}\rangle| &\le C\langle r\rangle^{1/2}\sum_{l,j=1}^{2}\|Q_j\|_{\dot X_{\zeta_j}^{-1/2}}\|Q_l\|_{\dot X_{\zeta_l}^{-1/2}}\\
&\quad+C\left(\tau^{-1}+\tau^{-\varepsilon/(1+\varepsilon)}+k\tau^{-1}\right)\langle r\rangle^{1/2}\|Q_1\|_{\dot X_{\zeta_1}^{-1/2}}\|Q_2\|_{\dot X_{\zeta_2}^{-1/2}}\\
&\quad+CA^{1/2}2\tau^2e^{2\sqrt{2}R\tau}.
\end{aligned}$$

Integrating on both sides and using Hölder inequality, we have

$$
\begin{aligned}
&|\langle Q_2 - Q_1 | e^{-ir\eta\cdot x}\rangle| \\
&\le C\langle r\rangle^{1/2} \sum_{l,j=1}^{2} \left(\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda} \|Q_l\|^2_{\dot{X}^{-1/2}_{\zeta_l}} d\tau d\eta_1\right)^{1/2} \left(\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda} \|Q_j\|^2_{\dot{X}^{-1/2}_{\zeta_j}} d\tau d\eta_1\right)^{1/2} \\
&\quad + C\left(\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda} \left(\tau^{-1} + \tau^{-\varepsilon/(1+\varepsilon)} + k\tau^{-1}\right)^2 \langle r\rangle \|Q_1\|^2_{\dot{X}^{-1/2}_{\zeta_1}} d\tau d\eta_1\right)^{1/2} \\
&\quad \times \left(\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda} \|Q_2\|^2_{\dot{X}^{-1/2}_{\zeta_2}} d\tau d\eta_1\right)^{1/2} + CA^{1/2}\lambda^2 e^{4\sqrt{2}R\lambda}.
\end{aligned}
$$

By the hypothesis $|\zeta| \ge \max\{1, C_* k\}$ in Lemma 5.1 and $|\zeta|^2 = 2\tau^2$, one has $2^{1/2}\tau = |\zeta| \ge \max\{1, C_* k\}$ which implies $(\tau^{-1} + \tau^{-\varepsilon/(1+\varepsilon)} + k\tau^{-1})$ is bounded by some constant which is independent of k. Thus, we deduce that the first and the second terms on the right hand side of the previous equation are both bounded by

$$
C\langle r\rangle^{1/2} \left(\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda} \|Q_l\|^2_{\dot{X}^{-1/2}_{\zeta_l}} d\tau d\eta_1\right).
$$

Applying Lemma 3.4, we then get

$$
|\langle Q_2 - Q_1 | e^{-ir\eta\cdot x}\rangle| \le C\langle r\rangle^{1/2} \left(\left(1 + \langle r\rangle^2 \lambda^{-1}\right)\lambda^{-\varepsilon/(1+\varepsilon)} + k^2\lambda^{-1}\right) + CA^{1/2}\lambda^2 e^{4\sqrt{2}R\lambda} \tag{5.6}
$$

with $r/2 \le \lambda$.

Denote that $\mathcal{F}(f)$ the Fourier transformation of a function f. We estimate $\|Q_2 - Q_1\|_{H^{-s}(\mathbb{R}^n)}$ by considering the low frequency and high frequency of the Fourier transform of $Q_2 - Q_1$. Thus we have the following result.

Proposition 5.2. *Let $2s > n+3$. Suppose that $a_0 \ge 2^{-1/2}C_*$ and $\lambda \ge a_0 k$ where C_* is the constant defined in Theorem 3.3. Then we have*

$$
\|Q_2 - Q_1\|_{H^{-s}(\mathbb{R}^n)} \le C\lambda^{-\varepsilon/(1+\varepsilon)} + Ck^2\lambda^{-1} + CA^{1/2}\lambda^2 e^{4\sqrt{2}R\lambda} + CkT^{-(s-1)}, \tag{5.7}
$$

where $A = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^2$ and C depends on $n, s, \varepsilon, \Omega$ and M.

Proof. Let $Q = Q_2 - Q_1$. Written in polar coordinates, we have that

$$
\begin{aligned}
\|Q\|^2_{H^{-s}(\mathbb{R}^n)} &\le C\int_0^\infty \int_{|\eta|=1} |\mathcal{F}Q(r\eta)|^2 \left(1+r^2\right)^{-s} r^{n-1} d\eta dr \\
&\le C\Big(\int_0^T \int_{|\eta|=1} |\mathcal{F}Q(r\eta)|^2 \left(1+r^2\right)^{-s} r^{n-1} d\eta dr \\
&\quad + \int_T^\infty \int_{|\eta|=1} |\mathcal{F}Q(r\eta)|^2 \left(1+r^2\right)^{-s} r^{n-1} d\eta dr\Big) \\
&=: C(I_1 + I_2),
\end{aligned}
$$

where T is a parameter which will be chosen later. We estimate I_2 first, then

$$\begin{aligned}
I_2 &\le C\int_{|\xi|\ge T}\left(1+|\xi|^2\right)^{-s}|\mathcal{F}Q(\xi)|^2d\xi\\
&= C\int_{|\xi|\ge T}\left(1+|\xi|^2\right)^{-s+1}\left(1+|\xi|^2\right)^{-1}|\mathcal{F}Q(\xi)|^2d\xi\\
&\le CT^{-2(s-1)}\|Q_2-Q_1\|^2_{H^{-1}(\mathbb{R}^n)}. \qquad (5.8)
\end{aligned}$$

To estimate $\|Q_2-Q_1\|^2_{H^{-1}(\mathbb{R}^n)}$. Recall that $Q_j = q_j + (D_j+ik)\gamma_j^{-1}$. Since for any $\phi\in H^1(\mathbb{R}^n)$,

$$\begin{aligned}
\langle q_j|\phi\rangle &= \int_{\mathbb{R}^n}\nabla\gamma_j^{1/2}\cdot\nabla\left(\gamma_j^{-1/2}\phi\right)dx\\
&= \int_{\mathbb{R}^n}\nabla\gamma_j^{1/2}\cdot\nabla\gamma_j^{-1/2}\phi+\gamma_j^{-1/2}\nabla\gamma_j^{1/2}\cdot\nabla\phi dx\\
&= \int_{\mathbb{R}^n}\mathcal{F}\left(\nabla\gamma_j^{1/2}\cdot\nabla\gamma_j^{-1/2}\right)\mathcal{F}\phi+\mathcal{F}\left(\gamma_j^{-1/2}\nabla\gamma_j^{1/2}\right)\cdot(i\xi)\mathcal{F}\phi d\xi,
\end{aligned}$$

one has

$$\begin{aligned}
\|q_j\|^2_{H^{-1}(\mathbb{R}^n)} &\le \int_{\mathbb{R}^n}(1+|\xi|^2)^{-1}\left|\mathcal{F}\left(\nabla\gamma_j^{1/2}\cdot\nabla\gamma_j^{-1/2}\right)+\mathcal{F}\left(\gamma_j^{-1/2}\nabla\gamma_j^{1/2}\right)\cdot(i\xi)\right|^2d\xi\\
&\le \|\nabla\gamma_j^{1/2}\cdot\nabla\gamma_j^{-1/2}\|^2_{L^2(\mathbb{R}^n)}+\|\gamma_j^{-1/2}\nabla\gamma_j^{1/2}\|^2_{L^2(\mathbb{R}^n)}\le C,
\end{aligned}$$

where C depends on n,Ω and M. Thus we obtain that if $k\ge 1$,

$$\|Q_2-Q_1\|^2_{H^{-1}(\mathbb{R}^n)}\le\|q_2-q_1\|^2_{H^{-1}(\mathbb{R}^n)}+\|\left(D_1+ik\right)\gamma_1^{-1}-\left(D_2+ik\right)\gamma_2^{-1}\|^2_{H^{-1}(\mathbb{R}^n)}\le Ck^2,$$

which gives

$$I_2\le Ck^2T^{-2(s-1)}$$

with C depends on n,Ω and M.

To estimate I_1, since $2s>n+3$, one has

$$\int_0^T\int_{|\eta|=1}\langle r\rangle^\delta(1+r^2)^{-s}r^{n-1}d\eta dr\le\int_0^\infty\int_{|\eta|=1}\langle r\rangle^\delta(1+r^2)^{-s}r^{n-1}d\eta dr\le C,\quad\text{for }\delta\le 3$$

which gives

$$\begin{aligned}
&\int_0^T\int_{|\eta|=1}\langle r\rangle\left(\left(1+\langle r\rangle^2\lambda^{-1}\right)^2\lambda^{-2\varepsilon/(1+\varepsilon)}+k^4\lambda^{-2}\right)(1+r^2)^{-s}r^{n-1}d\eta dr\\
&\le C\lambda^{-2\varepsilon/(1+\varepsilon)}+Ck^4\lambda^{-2}. \qquad (5.9)
\end{aligned}$$

Here we use the relation $r/2\le\lambda$ in (5.6) such that $\langle r\rangle\left(1+\langle r\rangle^2\lambda^{-1}\right)^2\le C\langle r\rangle\left(1+\langle r\rangle\right)^2\le C\langle r\rangle^3$. Using (5.6) and (5.9), it follows that

$$\begin{aligned}
I_1 &\le C\lambda^{-2\varepsilon/(1+\varepsilon)}+Ck^4\lambda^{-2}+C\int_0^T\int_{|\eta|=1}A\lambda^4e^{8\sqrt{2}R\lambda}(1+r^2)^{-s}r^{n-1}d\eta dr\\
&\le C\lambda^{-2\varepsilon/(1+\varepsilon)}+Ck^4\lambda^{-2}+CA\lambda^4e^{8\sqrt{2}R\lambda}. \qquad (5.10)
\end{aligned}$$

This completes the proof.

□

Now we prove the main result.

Proof of Theorem 2.1. From Proposition 5.2, we have

$$\|Q_2 - Q_1\|_{H^{-s}(\mathbb{R}^n)} \le C\lambda^{-\varepsilon/(1+\varepsilon)} + Ck^2\lambda^{-1} + CA^{1/2}\lambda^2 e^{4\sqrt{2}R\lambda} + CkT^{-(s-1)}.$$

Let $\alpha > 2$, taking $\lambda = T^\alpha$ such that λ satisfies $\lambda \ge T$ for $T \ge 1$. We deduce

$$\|Q_2 - Q_1\|_{H^{-s}(\mathbb{R}^n)} \le CT^{-\alpha\varepsilon/(1+\varepsilon)} + Ck^2T^{-\alpha} + CA^{1/2}T^{2\alpha}e^{4\sqrt{2}RT^\alpha} + CkT^{-(s-1)}. \tag{5.11}$$

We define $2E = \log\frac{1}{A}$ and $2c = 16\sqrt{2}R$ and consider the following two cases:

$$\text{(i) } c^{-1}E \ge a_0k^\alpha, \qquad \text{(ii) } a_0k^\alpha \ge c^{-1}E.$$

In case (i), let

$$T = (c^{-1}E)^{1/\alpha}, \tag{5.12}$$

such that

$$e^{4\sqrt{2}RT^\alpha} = A^{-1/4}.$$

We deduce from (5.11) that

$$\|Q_2 - Q_1\|_{H^{-s}(\mathbb{R}^n)} \le CT^{-\alpha\varepsilon/(1+\varepsilon)} + Ck^2T^{-\alpha} + CkT^{-(s-1)} + CT^{2\alpha}A^{1/4}. \tag{5.13}$$

Note that this choice of T is possible if

$$E \ge c,$$

that is,

$$A \le e^{-2c}$$

since T can not be smaller than 1.

Substituting $T = (c^{-1}E)^{1/\alpha}$ into (5.13), we obtain

$$\begin{aligned}
&\|Q_2 - Q_1\|_{H^{-s}(\mathbb{R}^n)} \\
&\le C(c^{-1}E)^{-\varepsilon/(1+\varepsilon)} + Ck^2(c^{-1}E)^{-1} + Ck(c^{-1}E)^{-(s-1)/\alpha} + C(c^{-1}E)^2A^{1/4} \\
&\le C\frac{(2c)^{\varepsilon/(1+\varepsilon)}}{(2E)^{\varepsilon/(1+\varepsilon)}} + C\frac{2ck^2}{2E} + C\frac{(2c)^{(s-1)/\alpha}k}{(2E)^{(s-1)/\alpha}} + C(c^{-1}E)^2A^{1/4} \\
&\le C\frac{(2c)^{\varepsilon/(1+\varepsilon)}}{(E+ca_0k^\alpha)^{\varepsilon/(1+\varepsilon)}} + C\frac{2ck^2}{E+ca_0k^\alpha} + C\frac{(2c)^{(s-1)/\alpha}k}{(E+ca_0k^\alpha)^{(s-1)/\alpha}} + CE^2A^{1/4} \qquad (5.14)
\end{aligned}$$

for $E \ge ca_0k^\alpha$.

On the other hand, in case (ii), the range $a_0k^\alpha \ge c^{-1}E$ implies the high frequency range of k. We substitute $T = a_0^{1/\alpha}k$ into (5.11) and use $a_0k^\alpha \ge c^{-1}E$, then

$$\begin{aligned}
&\|Q_2 - Q_1\|_{H^{-s}(\mathbb{R}^n)} \\
&\le C(a_0k^\alpha)^{-\varepsilon/(1+\varepsilon)} + Ck^2(a_0k^\alpha)^{-1} + CA^{1/2}a_0^2k^{2\alpha}e^{4\sqrt{2}Ra_0k^\alpha} + Ck(a_0k^\alpha)^{-(s-1)/\alpha} \\
&\le C(E+k^\alpha)^{-\varepsilon/(1+\varepsilon)} + Ck^2(E+k^\alpha)^{-1} + Ck(E+k^\alpha)^{-(s-1)/\alpha} + CA^{1/2}a_0^2k^{2\alpha}e^{4\sqrt{2}Ra_0k^\alpha}.
\end{aligned}$$

The proof is complete.

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