Inverse Problems and Imaging doi:10.3934/ipi.2011.5.xx
Volume 5, No. 1, 2011, 1–XX

GLOBAL UNIQUENESS FOR AN INVERSE PROBLEM FOR THE MAGNETIC SCHRÖDINGER OPERATOR

Ru-Yu Lai
Department of Mathematics, University of Washington
Seattle, Washington 98105, USA

(Communicated by Gunther Uhlmann)

Abstract. In this paper, we prove the global uniqueness of determining both the magnetic field and the electrical potential by boundary measurements in two-dimensional case. In other words, we prove the uniqueness of this inverse problem without any smallness assumption.

1. **Introduction.** Assume that Ω is a bounded domain in $\mathbb{R}^2$. In this article, we are interested in the global uniqueness of the inverse boundary value problem for two-dimensional Pauli Hamiltonian system:

$$H_{\overrightarrow{A},q}u := \sum_{j=1}^{2}(D_j - A_j)^2 u + rot\overrightarrow{A}u - qu = 0, \tag{1}$$

where $D_j = \frac{1}{i}\partial_j$ and $rot\overrightarrow{A} = \partial_1 A_2 - \partial_2 A_1$. Note that the real-valued vector field $\overrightarrow{A} = (A_1, A_2)$ is the magnetic potential, $rot\overrightarrow{A}$ is the magnetic field and q is the electrical potential.

The Cauchy data for the Pauli Hamiltonian (1) is

$$\begin{aligned} C_{\overrightarrow{A},q} = \{(f,g) : u|_{\partial\Omega} = f, \nu \cdot (\nabla u - i\overrightarrow{A}u)|_{\partial\Omega} = g, \\ u \in C^{1+\alpha}(\overline{\Omega}) \text{ is a solution of (1)}\} \end{aligned} \tag{2}$$

Assume that 0 is not the Dirichlet eigenvalue for $H_{\overrightarrow{A},q}$, then we can define the Dirichlet-to-Neumann(DN) operator by

$$\Lambda_{\overrightarrow{A},q} = \frac{\partial u}{\partial \nu}\big|_{\partial\Omega} - i(\overrightarrow{A}\cdot\nu)u\big|_{\partial\Omega}.$$

It is well-known that any gauge transformation $\overrightarrow{A} \to \overrightarrow{A} + \nabla\phi$ can deduce that $\Lambda_{\overrightarrow{A},q} = \Lambda_{\overrightarrow{A}+\nabla\phi,q}$ where $\phi \in C^1(\overline{\Omega}), \phi|_{\partial\Omega}$ and $\nabla\phi|_{\partial\Omega} = 0$. Therefore, the best we can do is recover the magnetic field and electrical potential from the DN map.

In this article, we consider an inverse problem for the Pauli Hamiltonian. We want to determine $rot\overrightarrow{A}$ and q from the boundary measurement. For the dimensions $n \geq 3$, an important outcome is given by Sun in 1993 with some smallness assumption, that is, $rot\overrightarrow{A}$ is small and q is in an open and dense set in some topology. Later in 1995, the global uniqueness had been solved by Nakamura, Sun and Uhlmann under the smoothness assumption on conductivities and boundary. In 1998, Tolmasky [25] relaxed regularity for $\overrightarrow{A}_j \in C^1_{\Omega}$. Salo [19] also reduced the

Key words and phrases. Dirichlet-to-Neumann map, magnetic fiel, electrical potential, Cauchy data, the Pauli Hamiltonian.

smoothness assumption to Dini continuous. Moreover, Salo [18] also presented a constructive result for magnetic field and electrical potential by applying the methods in Nachman [14].

The first result for two-dimensional case was presented by Sun [21]. The magnetic field and electrical potential are uniquely determined under the assumption $rot\overrightarrow{A}$ is small and q is in an open and dense set in some topology. In [12], Kang and Uhlmann applied the result in [7] and used the scattering transform in [4] [5] to determine the magnetic field and electrical potential from the Cauchy data of the Pauli Hamiltonian with $\| q_1 \|_{W^{1,p}(\Omega)} \leq \varepsilon$. Recently, in 2008, in Bukhgeim's paper [3], the potential is uniquely determined from the Cauchy data without any smallness assumption. In this article, with the help of Bukhgeim's result in [3], we remove the smallness assumption on the electrical potential in the work of Kang and Uhlmann [12].

In Bukhgeim's paper [3], the potential is uniquely determined by the set of Cauchy data for the equation.

$$\Delta u + au = 0 \tag{3}$$

in two dimensions without any smallness assumption. Moreover, he considered a first order system instead of (3). This result inspires us to use the second order equation which is described in [12] and to get the related first order system. Then, by using Bukhgeim's result, we can determine $rot\overrightarrow{A}$ and q from Cauchy data uniquely.

For $m \in \mathbb{Z}$, we denote

$$C^m_\Omega = \{f \in C^m(\mathbb{R}^2); Supp f \subset \Omega\} \text{ and } L^\infty_\Omega = \{f \in L^\infty(\mathbb{R}^2); Supp f \subset \Omega\}. \tag{4}$$

The following is our main result:

Theorem 1.1. *Let $\overrightarrow{A}_j \in C^1_\Omega(\mathbb{R}^2;\mathbb{R}^2)$ and $q_j \in C^1_\Omega(\mathbb{R}^2;\mathbb{R})$ for $j = 1, 2$. If*

$$C_{\overrightarrow{A}_1,q_1} = C_{\overrightarrow{A}_2,q_2},$$

then

$$rot\overrightarrow{A}_1 = rot\overrightarrow{A}_2 \quad and \quad q_1 = q_2 \quad in \quad \Omega. \tag{5}$$

In section 2, we will derive a second order equation from the Pauli Hamiltonian and introduce a result for this second order equation. About section 3, we construct CGO solutions and deduce some properties for CGO solutions. Moreover, we get boundary conditions from the Cauchy data through the ideas in [18] and [21]. For section 4, the main theorem is proved in this section by applying Bukhgeim's result. In section 5, we give formulas for the magnetic field and the electric potential by using the method given in [3].

2. **A second order equation.** Here is the second order equation we reduced from the Pauli Hamiltonian: Let $\overline{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$ and $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$. Computing this equation

$$H_{\overrightarrow{A},q}u = \sum_{j=1}^{2}(D_j - A_j)^2 u + rot\overrightarrow{A}u - qu = 0, \tag{6}$$

then we get

$$\Delta u - iA \cdot \nabla u - i\nabla \cdot (Au) - (|A|^2 + rot\overrightarrow{A} - q)u = 0. \tag{7}$$

The above equation implies

$$4(\overline{\partial}+\overline{a})(\partial-a)u+qu=0, \tag{8}$$

where $a=\frac{1}{2}(A_2+iA_1)$. Hence, we obtain the second order equation

$$(\overline{\partial}+\overline{a})(\partial-a)u-q'u=0, \tag{9}$$

where $q'=-\frac{1}{4}q$. The Cauchy data for the equation (9) is defined by

$$\begin{aligned} C_{a,q'} = \{(f,g) : u|_{\partial\Omega} = f, (\partial-a)u|_{\partial\Omega}=g, \\ u\in C^{1+\alpha}(\overline{\Omega}) \text{ is a solution of } (9)\} \end{aligned} \tag{10}$$

Definition 2.1. We define the operator $\overline{\partial}^{-1}$ by

$$\overline{\partial}^{-1}f(z)=-\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{f(\zeta)}{\zeta-z}d\zeta. \tag{11}$$

Similarly, ∂^{-1} is defined by

$$\partial^{-1}f(z)=-\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{f(\zeta)}{\overline{\zeta-z}}d\zeta. \tag{12}$$

Then we introduce the theorem for the second order equation.

Theorem 2.2. *Let $a_j\in C^1_\Omega(\mathbb{R}^2;\mathbb{C})$ and $q_j\in L^\infty_\Omega(\mathbb{R}^2;\mathbb{R})$, $j=1,2$. If $C_{a_1,q_1}=C_{a_2,q_2}$, then we obtain*

$$q_1=q_2 \quad \textit{and} \quad \overline{\partial}^{-1}\overline{a}_1+\partial^{-1}a_1=\overline{\partial}^{-1}\overline{a}_2+\partial^{-1}a_2 \quad \textit{in} \quad \Omega \tag{13}$$

In Section 4, we are going to see that theorem 1.1 is a direct consequence of Theorem 2.2.

A particular case of Theorem 1.1 is when the magnetic field is zero. That is, $rot\overrightarrow{A}_j=0,\ j=1,2$. Thus we have the equation

$$\sum_{j=1}^{2}(D_j-A_j)^2u-qu=0 \tag{14}$$

and get the following result:

Corollary 1. *Let $\overrightarrow{A}_j\in C^1_\Omega(\mathbb{R}^2;\mathbb{R}^2)$ and $q_j\in C^1_\Omega(\mathbb{R}^2;\mathbb{R})$, $j=1,2$. If*

$$C_{\overrightarrow{A}_1,q_1}=C_{\overrightarrow{A}_2,q_2}$$

then

$$q_1=q_2 \quad \textit{in} \quad \Omega \tag{15}$$

Remark 1. The above corollary implies tat Cauchy data determines the electrical potential q uniquely when $rot\overrightarrow{A}_j=0,\ j=1,2$. Furthermore, in corollary 1, $\overrightarrow{A}_j,\ j=1,2$ only need one-derivative. This result is different from the one presented by Sun in two-dimensional case.

3. **Construction of CGO solutions.** The Pauli Hamiltonian in two dimensions is

$$H_{\overrightarrow{A},q}u = \sum_{j=1}^{2}(D_j - A_j)^2 u + rot\overrightarrow{A}u - qu = 0. \tag{16}$$

We look for the complex geometrical optics (CGO) solutions to (16) in $\mathbb{R}^2$. We consider the solutions of the form

$$u(x,\rho) = e^{\rho\cdot x}(e^{\phi(x)} + \omega(x,\rho)), \tag{17}$$

where $\rho \in \mathbb{C}^2$ with $\rho\cdot\rho = 0$ and ω is decaying in $|\rho|$. Substituting the above solutions into equation (16), then it deduces the following two equations:

$$\rho\cdot\nabla\phi = i\rho\cdot\overrightarrow{A}; \tag{18}$$

$$\begin{aligned}&(-\Delta - 2i\rho\cdot\nabla + 2i\overrightarrow{A}\cdot\nabla - 2\rho\cdot\overrightarrow{A} + G)\omega\\ &= -(-\Delta - 2i\rho\cdot\nabla + 2i\overrightarrow{A}\cdot\nabla - 2\rho\cdot\overrightarrow{A} + G)e^{\phi},\end{aligned} \tag{19}$$

where $G = \overrightarrow{A}^2 - D\cdot\overrightarrow{A} + rot\overrightarrow{A} - q$. Let $\Delta_\rho = -\Delta - 2i\rho\cdot\nabla$ and $\nabla_\rho = -i\nabla + \rho$. Then (19) can be rewritten as

$$(\Delta_\rho - 2\overrightarrow{A}\cdot\nabla_\rho + G)\omega = -f, \tag{20}$$

where $f = (\Delta_\rho - 2\overrightarrow{A}\cdot\nabla_\rho + G)e^{\phi}$.

Denote $L^2_{\delta+1}(\mathbb{R}^n)$ be the weighted L^2 space where $\delta\in\mathbb{R}$. That is, for $f\in L^2_{\delta+1}(\mathbb{R}^n)$, it means $\langle x\rangle^{\delta+1}f \in L^2(\mathbb{R}^n)$ where $\langle x\rangle^{\delta+1} = (1+|x|^2)^{\frac{\delta+1}{2}}$. Furthermore, Ω^c represents the complement of Ω.

By using the ideas in [21], then we have the following special case.

Lemma 3.1. *Considering $\rho = t\zeta$, where $\zeta = (1,i)$ for $t\in\mathbb{R}-\{0\}$. Then*

$$\overline{\partial}\phi = -\overline{a}. \tag{21}$$

Proof. We have decided the direction of ρ. Thus, from (18), it deduces that

$$(1,i)\cdot\nabla\phi = i(1,i)\cdot\overrightarrow{A},$$

which implies

$$(\partial_1 + i\partial_2)\phi = -(A_2 - iA_1).$$

This leads to (21). □

The following lemma is a special case in [17], Theorem 2.1 (see [24] for a similar estimate).

Lemma 3.2. *(Sylvester and Uhlmann, 1986). Let $-1<\delta<0$. For $f\in L^2_\delta(\mathbb{R}^2)$, there exists a unique solution $u\in L^2_\delta(\mathbb{R}^2)$ of $\overline{\partial}u = f$. Moreover,*

$$u(z) = -\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{f(\zeta)}{\zeta - z}d\zeta \tag{22}$$

and

$$\|u\|_{L^2_\delta} + \|\nabla u\|_{L^2_{\delta+1}} \le C\|f\|_{L^2_{\delta+1}}. \tag{23}$$

By this result, we can deduce the following lemma as in [21].

Lemma 3.3. *Let $a \in C^1_\Omega(\mathbb{R}^2;\mathbb{C})$. There exists a unique solution ϕ in $H^2_\delta(\mathbb{R}^2)$ such that $\overline{\partial}\phi = -\overline{a}$. Moreover,*

$$\|\phi\|_{L^2_\delta} + \|\nabla\phi\|_{H^1_{\delta+1}} \le C\|\overline{a}\|_{H^1(\Omega)} \tag{24}$$

and

$$|\phi(z)| \le C\|\overline{a}\|_{L^\infty(\Omega)} \quad \textit{for all } z \in \mathbb{R}^2. \tag{25}$$

Proof. Since $a \in C^1_\Omega(\mathbb{R}^2;\mathbb{C})$, we have $a \in L^2_\delta$. By lemma 3.2, it deduces that there exists a unique solution, say ϕ, satisfies $\overline{\partial}\phi = -\overline{a}$. And ϕ can be written as

$$\phi(z) = \frac{1}{\pi}\int_\Omega \frac{\overline{a}(\zeta)}{\zeta - z} d\zeta$$

and

$$\|\phi\|_{L^2_\delta} + \|\nabla\phi\|_{L^2_{\delta+1}} \le C\|\overline{a}\|_{L^2_{\delta+1}}.$$

Moreover, $\overline{\partial}\big(\frac{\partial^\alpha}{\partial x^\alpha}\phi\big) = -\frac{\partial^\alpha}{\partial x^\alpha}\overline{a}$ for $\alpha = (\alpha_1, \alpha_2)$ and $|\alpha| = 1$. Again, by lemma 3.2, we get

$$\|\frac{\partial^\alpha}{\partial x^\alpha}\phi\|_{L^2_\delta} + \|\nabla\big(\frac{\partial^\alpha}{\partial x^\alpha}\phi\big)\|_{L^2_{\delta+1}} \le C\|\frac{\partial^\alpha}{\partial x^\alpha}\overline{a}\|_{L^2_{\delta+1}} \quad \text{for} \quad |\alpha| = 1.$$

Now, computing

$$\begin{aligned} &\|\phi\|_{L^2_\delta} + \|\nabla\phi\|_{H^1_{\delta+1}} \\ \le&\|\phi\|_{L^2_\delta} + \sum_{1\le|\alpha|\le 2}\|\big(\frac{\partial^\alpha}{\partial x^\alpha}\phi\big)\|_{L^2_{\delta+1}} \\ \le&C\sum_{0\le|\alpha|\le 1}\|\frac{\partial^\alpha}{\partial x^\alpha}\overline{a}\|_{L^2_{\delta+1}}. \end{aligned} \tag{26}$$

Because $\overline{a}$ has compact support, we get

$$\|\frac{\partial^\alpha}{\partial x^\alpha}\overline{a}\|_{L^2_{\delta+1}} \le C\|\frac{\partial^\alpha}{\partial x^\alpha}\overline{a}\|_{L^2} \quad \text{for} \quad 0 \le |\alpha| \le 1. \tag{27}$$

From (26) and (27), the inequality (24) follows.

To show that $\|\phi\|_{L^\infty} < \infty$. Since Ω is bounded, we can find a ball $B = B(0, M)$ with center 0 and radius M with $0 < M < \infty$ such that $B \supseteq \Omega$. Then we discuss the following two cases:
If $|z| \ge 2M$, then $|\zeta - z| \ge M$ for $\zeta \in B$ and

$$\begin{aligned} |\phi(z)| \le&\big|\frac{1}{\pi}\int_\Omega \frac{\overline{a}(\zeta)}{\zeta - z} d\zeta\big| \\ \le&\frac{1}{\pi}\|\overline{a}\|_{L^\infty}\int_B \frac{1}{|\zeta - z|} d\zeta \\ \le&\frac{1}{\pi}\|\overline{a}\|_{L^\infty}\int_B \frac{1}{M} d\zeta \\ =&M\|\overline{a}\|_{L^\infty}. \end{aligned}$$

For $|z| \leq 2M$, let $\eta = \zeta - z$ for $\zeta \in B$, then $|\eta| \leq |\zeta| + |z| \leq M + 2M$. Therefore, we obtain

$$\begin{aligned}|\phi(z)| \leq & \frac{1}{\pi}\|\overline{a}\|_{L^\infty} \int_B \frac{1}{|\zeta - z|} d\zeta \\ \leq & \frac{1}{\pi}\|\overline{a}\|_{L^\infty} \int_{B(0,3M)} \frac{1}{|\eta|} d\eta \\ = & 6M\|\overline{a}\|_{L^\infty}.\end{aligned}$$

Taking $C = 6M$, so $|\phi(z)| \leq C\|\overline{a}\|_{L^\infty(\Omega)}$ for all $z \in \mathbb{R}^2$. That is, $\|\phi\|_{L^\infty} < \infty$. □

To construct the CGO solutions, it is important to know whether the equation (20) has a unique solution under some requirements. From [18], Salo proved the uniqueness and existence of (20) when $W \in C_c(\mathbb{R}^2; \mathbb{C}^2)$ and $\tilde{q} \in L_c^\infty(\mathbb{R}^2; \mathbb{C})$. Here is Salo's result in [18].

Lemma 3.4. *(Salo, 2006) Let $W \in C_c(\mathbb{R}^2; \mathbb{C}^2)$ and $q \in L_c^\infty(\mathbb{R}^2; \mathbb{C})$ and $-1 < \delta < 0$. If $\rho \in \mathbb{C}^2$ with $\rho \cdot \rho = 0$ and $|\rho|$ is large enough. For any $f \in L^2_{\delta+1}(\mathbb{R}^2)$, then there exists a unique solution $u \in H^1_\delta(\mathbb{R}^2)$ of the equation*

$$(\Delta_\rho + 2W \cdot \nabla_\rho + q)u = f. \tag{28}$$

Moreover, $u \in H^2_\delta(\mathbb{R}^2)$ and for $0 \leq s \leq 2$, u satisfies

$$\|u\|_{H^s_\delta} \leq C|\rho|^{s-1}\|f\|_{L^2_{\delta+1}},$$

where C is independent of ρ and f.

With the above lemma, we show that $H_{\overrightarrow{A},q}u = 0$ has the CGO solutions with specific form.

Theorem 3.5. *Let $\rho = t(1, i)$, $t \in \mathbb{R}$ and $-1 < \delta < 0$. Suppose that $\overrightarrow{A} \in C^1_\Omega(\mathbb{R}^2; \mathbb{R}^2)$ and $\rho \in L^\infty_\Omega(\mathbb{R}^2; \mathbb{R})$. For $|\rho|$ large enough, there exists a unique CGO solution u of $H_{\overrightarrow{A},q}u = 0$ with the form*

$$u(x, \rho) = e^{\rho \cdot x}(e^{\phi(x)} + \omega(x, \rho)), \tag{29}$$

where ω is the solution of (20) and is decaying in $|\rho|$, that is ,

$$\|\omega\|_{H^s_\delta} \leq C|\rho|^{s-1}\|f\|_{L^2_{\delta+1}} \quad \text{for } 0 \leq s \leq 2. \tag{30}$$

Proof. For any $\rho \in \mathbb{C}^2$ with $\rho \cdot \rho = 0$. Considering this kind of solutions

$$u(x, \rho) = e^{\rho \cdot x}(e^{\phi(x)} + \omega(x, \rho)), \tag{31}$$

to the equations $H_{\overrightarrow{A},q}u = 0$. Thus, it can deduces that

$$\rho \cdot \nabla\phi = i\rho \cdot \overrightarrow{A}, \tag{32}$$

$$(\Delta_\rho - 2\overrightarrow{A} \cdot \nabla_\rho + G)\omega = -f \tag{33}$$

where $G = \overrightarrow{A}^2 - D \cdot \overrightarrow{A} + rot\overrightarrow{A} - q \in L_c^\infty(\mathbb{R}^2)$ and $f = (\Delta_\rho - 2\overrightarrow{A} \cdot \nabla_\rho + G)e^\phi$.

To show $f \in L^2_{\delta+1}(\mathbb{R}^2)$. First, we take $\rho = t(1, i)$ where $t \in \mathbb{R}$, then (32) can be rewritten as

$$\overline{\partial}\phi = -\overline{a},$$

where $\overline{a} = \frac{A_2 - iA_1}{2}$ with $\overrightarrow{A} = (A_1, A_2)$. Since $\overrightarrow{A} \in C^1_\Omega(\mathbb{R}^2;\mathbb{R})$, we obtain $\overline{a} \in C^1_\Omega(\mathbb{R}^2;\mathbb{C})$. From lemma 3.3, it implies that we can find unique solution $\phi \in H^2_\delta(\mathbb{R}^2)$ of $\overline{\partial}\phi = -\overline{a}$ with the form

$$\phi(z) = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\overline{a}(\zeta)}{\zeta - z} d\zeta. \tag{34}$$

Furthermore, ϕ satisfies the inequalities

$$\|\phi\|_{L^2_\delta} + \|\nabla\phi\|_{H^1_{\delta+1}} \leq C\|\overline{a}\|_{H^1(\Omega)} \tag{35}$$

and

$$|\phi(x)| \leq C\|\overline{a}\|_{L^\infty(\Omega)} \quad \text{for all } x \in \mathbb{R}^2. \tag{36}$$

Computing

$$f(x) =(\Delta_\rho - 2\overrightarrow{A}\cdot\nabla_\rho + G)e^{\phi(x)} \tag{37}$$

$$=e^{\phi(x)}(-\Delta\phi - |\nabla\phi|^2 - 2i\rho\cdot\nabla\phi - 2\rho\cdot\overrightarrow{A} + 2i\overrightarrow{A}\cdot\nabla\phi + G)(x). \tag{38}$$

To estimate $\|e^\phi\|_{L^\infty}$, we get

$$\|e^\phi\|_{L^\infty} \leq e^{\|\phi\|_{L^\infty}} \tag{39}$$

$$\leq e^{C\|\overline{a}\|_{L^\infty}} \tag{40}$$

$$<\infty. \tag{41}$$

Since $\rho\cdot\nabla\phi = i\rho\cdot\overrightarrow{A}$, the $L^2_{\delta+1}$ norm for f is bounded by

(42)

$$\|f\|_{L^2_{\delta+1}} = \|e^{\phi(x)}\|_{L^\infty}\big(\|\Delta\phi\|_{L^2_{\delta+1}} + \||\nabla\phi|^2\|_{L^2_{\delta+1}} + \|2i\overrightarrow{A}\cdot\nabla\phi\|_{L^2_{\delta+1}} + \|G\|_{L^2_{\delta+1}}\big).$$

From the fact that $G \in L^\infty_c(\mathbb{R}^2)$, it deduces $G \in L^2_{\delta+1}$. Moreover, by (35), we have $\|\Delta\phi\|_{L^2_{\delta+1}} < \infty$. We consider

$$\begin{aligned}\||\nabla\phi|^2\|^2_{L^2_{\delta+1}} =& \int_{\mathbb{R}^2}(1+|x|^2)^{\delta+1}|\nabla\phi|^4 dx\\ \leq& C\int_{\mathbb{R}^2}(1+|x|^2)^{\delta+1}|\nabla\phi|^2|\nabla\overline{a}|^2 dx\\ \leq& \|\nabla\overline{a}\|^2_{L^\infty}\int_{\mathbb{R}^2}(1+|x|^2)^{\delta+1}|\nabla\phi|^2 dx\\ \leq& \|\nabla\overline{a}\|^2_{L^\infty}\|\nabla\phi\|^2_{L^2_{\delta+1}}\end{aligned}$$

and

$$\begin{aligned}\|2i\overrightarrow{A}\cdot\nabla\phi\|^2_{L^2_{\delta+1}} =& \int_{\mathbb{R}^2}(1+|x|^2)^{\delta+1}|\overrightarrow{A}|^2|\nabla\phi|^2 dx\\ \leq& \|\overrightarrow{A}\|^2_{L^\infty}\int_{\Omega}(1+|x|^2)^{\delta+1}|\nabla\phi|^2 dx\\ \leq& \|\overrightarrow{A}\|^2_{L^\infty}\|\nabla\phi\|^2_{L^2_{\delta+1}}.\end{aligned}$$

So, $\||\nabla\phi|^2\|_{L^2_{\delta+1}}$ and $\|2i\overrightarrow{A}\cdot\nabla\phi\|_{L^2_{\delta+1}}$ are finite. Therefore, f is in $L^2_{\delta+1}$.

Thus, from lemma 3.4, for $|\rho|$ large enough, there exists a unique solution $\omega \in H^1_\delta(\mathbb{R}^2)$ of the equation

$$(\Delta_\rho - 2\overrightarrow{A}\cdot\nabla_\rho + G)\omega = -f, \tag{43}$$

where ω satisfies

$$\|\omega\|_{H^s_\delta} \leq C|\rho|^{s-1}\|f\|_{L^2_{\delta+1}} \quad \text{for} \quad 0 \leq s \leq 2. \tag{44}$$

To show the form(31) for u is the CGO solution. We follow similar argument to the ones given in [18]. Considering

$$u(x,\rho) = e^{\rho\cdot x}(1+\tilde{\omega})$$

where $\tilde{\omega} = e^{\phi(x)} - 1 + \omega(x,\rho)$. Since $e^{\phi(x)} - 1$ and ω are in H^1_δ, we obtain $\tilde{\omega} \in H^1_\delta$. Moreover, $\tilde{\omega}$ satisfies this equation

$$(\Delta_\rho - 2\overrightarrow{A}\cdot\nabla_\rho + G)\omega = -(G - 2\overrightarrow{A}\cdot\rho), \tag{45}$$

where $G - 2\overrightarrow{A}\cdot\rho = \overrightarrow{A}^2 - D\cdot\overrightarrow{A} + rot\overrightarrow{A} - q - 2\overrightarrow{A}\cdot\rho \in L^\infty_c(\mathbb{R}^2)\cap L^2_{\delta+1}(\mathbb{R}^2)$. Therefore, from lemma 3.4, $\tilde{\omega}$ is unique and has the properties

$$\|\tilde{\omega}\|_{H^s_\delta} \leq C|\rho|^{s-1}\|G - 2\overrightarrow{A}\cdot\rho\|_{L^2_{\delta+1}} \quad \text{for} \quad 0 \leq s \leq 2. \tag{46}$$

Hence, u is the unique solution of $H_{\overrightarrow{A},q}u = 0$ with the form $u(x,\rho) = e^{\rho\cdot x}(e^{\phi(x)} + \omega(x,\rho))$. □

With the help of the above lemma, it provides us a key to get the boundary information of $\overline{\partial}^{-1}\overline{a}$.

Lemma 3.6. *Let $\overrightarrow{A}j \in C^1_\Omega(\mathbb{R}^2;\mathbb{R}^2)$ and $q_j \in L^\infty_\Omega(\mathbb{R}^2;\mathbb{R})$, $j = 1,2$. If $C_{a_1,q_1} = C_{a_2,q_2}$, then $\overline{\partial}^{-1}\overline{a}_1 = \overline{\partial}^{-1}\overline{a}_2$ in Ω^c.*

Proof. From theorem 3.5, for any $\rho = t(1,i) \in \mathbb{C}^2$, $t \in \mathbb{R}$. Considering the CGO solutions

$$u_j(x,\rho) = e^{\rho\cdot x}(e^{\phi_j(x)} + \omega_j(x,\rho)), \quad j = 1,2 \tag{47}$$

to the equation $H_{\overrightarrow{A_j},q_j}u = 0$, respectively. Thus, it can deduces that

$$\overline{\partial}\phi_j = -\overline{a}_j, \tag{48}$$

$$(\Delta_\rho - 2\overrightarrow{A}_j\cdot\nabla_\rho + G_j)\omega_j = -f_j, \tag{49}$$

where $G_j = \overrightarrow{A}_j^2 - D\cdot\overrightarrow{A}_j + rot\overrightarrow{A}_j - q_j \in L^\infty_c(\mathbb{R}^2)$ and $f_j = (\Delta_\rho - 2\overrightarrow{A}_j\cdot\nabla_\rho + G_j)e^{\phi_j}$. Moreover,

$$\|\omega_j\|_{H^s_\delta} \leq C|\rho|^{s-1}\|f_j\|_{L^2_{\delta+1}} \quad \text{for} \quad 0 \leq s \leq 2. \tag{50}$$

Following similar argument to the ones given in [21], we can prove that

$$\phi_1 = \phi_2 \quad \text{in } \Omega^c. \tag{51}$$

Before proving (51),we first show the following identity:

$$u_1(x,\rho) = u_2(x,\rho) \quad \text{for } x \in \Omega^c \text{ and } |\rho| \text{ large enough.} \tag{52}$$

Let u be a solution of the Dirichlet problem

$$H_{\overrightarrow{A}_2,q_2}u = 0 \text{ in } \Omega;$$
$$u|_{\partial\Omega} = u_1|_{\partial\Omega}.$$

Denote the function u_3 by

$$u_3 = \begin{cases} u, & \text{in } \Omega; \\ u_1, & \text{in } \Omega^c. \end{cases}$$

Since $C_{a_1,q_1} = C_{a_2,q_2}$, it implies u_3 is a solution of $H_{\overrightarrow{A}_2,q_2} u = 0$ in $\mathbb{R}^2$. Thus u_3 can be written as

$$u_3(x,\rho) = e^{\rho\cdot x}(e^{\phi_2(x)} + \omega_3(x,\rho)).$$

Hence, ω_3 must satisfies

$$(\Delta_\rho - 2\overrightarrow{A}_2 \cdot \nabla_\rho + G_2)\omega = -f_2. \tag{53}$$

Suppose that we can show that $\omega_3 \in H^1_\delta(\mathbb{R}^2)$. We have known that $\omega_2 \in H^1_\delta(\mathbb{R}^2)$ is also a solution of equation (53). From the lemma 3.4, we know that there exists a unique solution in $H^1_\delta(\mathbb{R}^2)$ of equation (53) if $|\rho|$ is large enough, then it can deduce $\omega_3 = \omega_2$. Therefore, we obtain $u_3 = u_2$ in $\mathbb{R}^2$. Since $u_3 = u_1$ in Ω^c, we have $u_1 = u_2$ in Ω^c for $|\rho|$ large enough.

Now, we start to show $\omega_3 \in H^1_\delta(\mathbb{R}^2)$. Considering

$$e^{\rho\cdot x}(e^{\phi_1(x)} + \omega_1(x,\rho)) = e^{\rho\cdot x}(e^{\phi_2(x)} + \omega_3(x,\rho)) \text{ in } \Omega^c,$$

which implies

$$\omega_3 = e^{\phi_1} - e^{\phi_2} + \omega_1.$$

Then

$$|\omega_3| \le |\phi_1| + |\phi_2| + |\omega_1|.$$

Since ϕ_1, ϕ_2 and ω_1 are in $H^1_\delta(\mathbb{R}^2)$, we get $\omega_3 \in H^1_\delta(\mathbb{R}^2)$.

By (44) and (52), it follows that $\phi_1 = \phi_2$ in Ω^c, as $|\rho| \to \infty$. From lemma 3.2, ϕ_j can be expressed as

$$\phi_j(z) = \frac{1}{\pi}\int_\Omega \frac{\overline{a}_j(\zeta)}{\zeta - z} d\zeta \text{ for } j = 1,2. \tag{54}$$

Since $\phi_1 = \phi_2$ in Ω^c, then it implies

$$\frac{1}{\pi}\int_\Omega \frac{(\overline{a}_1 - \overline{a}_2)(\zeta)}{\zeta - z} d\zeta = 0 \text{ in } \Omega^c.$$

Thus

$$\overline{\partial}^{-1}\overline{a}_1 = \overline{\partial}^{-1}\overline{a}_2 \text{ in } \Omega^c.$$

□

4. **Proof of the main theorem.** In this section, we firstly introduce the uniqueness identifiability for a first order system which is proven by Bukhgeim [3]. Later, we reduce the Pauli Hamiltonian to a second order equation and obtain a similar first order system as the one in [3]. Let

$$\tilde{D} = \begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} \;;\; B = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$$

where a, b are complex functions. Denote

$$P := \tilde{D} - B. \tag{55}$$

The set of Cauchy data for $Pu = 0$ is defined by

$$C_B = \{u|_{\partial\Omega} : Pu = 0,\ u \in C^{1+\alpha}(\overline{\Omega})\}. \tag{56}$$

In [3], Bukhgeim obtained the following result.

Theorem 4.1. *(Bukhgeim, 2008) If $B_j \in C^1(\Omega)$, $j = 1,2$ and $C_{B_1} = C_{B_2}$, then $B_1 = B_2$.*

Now, we are going to find similar form like (55). In section 2, we have deduced the following second order equation

$$(\overline{\partial}+\overline{a})(\partial-a)u-qu=0 \ \text{ in } \Omega \tag{57}$$

from the Pauli Hamiltonian, where $a=\frac{1}{2}(A_2+iA_1)$. The Cauchy data for (57) is

$$\begin{aligned} C_{a,q}=\{(f,g): u|_{\partial\Omega}=f, (\partial-a)u|_{\partial\Omega}=g, \\ u\in C^{1+\alpha}(\overline{\Omega}) \text{ is a solution of } (57)\} \end{aligned} \tag{58}$$

Let $\omega:=(\partial-a)u$, then by (57) we obtain

$$\left[\begin{pmatrix} \overline{\partial}+\overline{a} & 0 \\ 0 & \partial-a \end{pmatrix} - \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} \omega \\ u \end{pmatrix}=0. \tag{59}$$

Define

$$Ta(z):=(\overline{\partial}^{-1}\overline{a}+\partial^{-1}a)(z) \ \text{ and } \ F=\begin{pmatrix} e^{\overline{\partial}^{-1}\overline{a}} & 0 \\ 0 & e^{-\partial^{-1}a} \end{pmatrix},$$

then we rewrite (59) to the form

$$F^{-1}\left[\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} - \begin{pmatrix} 0 & e^{Ta}q \\ e^{-Ta} & 0 \end{pmatrix}\right]F\begin{pmatrix} \omega \\ u \end{pmatrix}=0. \tag{60}$$

Denote

$$P:=\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} - \begin{pmatrix} 0 & e^{Ta}q \\ e^{-Ta} & 0 \end{pmatrix}=\tilde{D}-Q, \tag{61}$$

where

$$\tilde{D}=\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} \ \text{ and } \ Q=\begin{pmatrix} 0 & e^{Ta}q \\ e^{-Ta} & 0 \end{pmatrix}.$$

Therefore, we obtain te first order system which is similar to the one in [3]. This Cauchy data for $P\tilde{u}=0$ is also defined by

$$C_Q=\{\tilde{u}|_{\partial\Omega}: P\tilde{u}=0, \ \tilde{u}\in C^{1+\alpha}(\overline{\Omega})\}. \tag{62}$$

The following is the proof of Theorem 2.2.

Proof of Theorem 2.2. The first step is to prove that $C_{Q_1}=C_{Q_2}$. Suppose $v_1\in C_{Q_1}$, then it satisfies

$$(\tilde{D}-Q_1)v_1=0.$$

Let $\tilde{u}_1=F_1^{-1}v_1$ where F_1^{-1} is invertible. Thus $\tilde{u}_1$ is a solution of

$$F_1^{-1}(\tilde{D}-Q_1)F_1u=0.$$

Since $C_{a_1,q_1}=C_{a_2,q_2}$, there exists a solution $\tilde{u}_2$ of

$$F_2^{-1}(\tilde{D}-Q_2)F_2u=0$$

such that

$$\tilde{u}_1|_{\partial\Omega}=\tilde{u}_2|_{\partial\Omega}. \tag{63}$$

Let $v_2=F_2\tilde{u}_2$. Because F_2 is invertible, it deduces that v_2 is a solution of

$$(\tilde{D}-Q_2)v=0$$

and then $v_2\in C_{Q_2}$. Since $\partial^{-1}a_1=\partial^{-1}a_2$ on $\partial\Omega$, we have $F_1=F_2$ on $\partial\Omega$. By (63), it obtains

$$v_1|_{\partial\Omega}=(F_1\tilde{u}_1)|_{\partial\Omega}=(F_2\tilde{u}_2)|_{\partial\Omega}=v_2|_{\partial\Omega}.$$

Consequently, $C_{Q_1} = C_{Q_2}$. Using Theorem 4.1, we conclude that

$$Q_1 = Q_2 \text{ in } \Omega.$$

That is,

$$e^{-Ta_1} = e^{-Ta_2};\ e^{Ta_1} q_1 = e^{Ta_2} q_2 \text{ in } \Omega.$$

Hence, $Ta_1 = Ta_2$ and $q_1 = q_2$ in Ω. From the definitions of $Ta_j,\ j = 1, 2$, it implies $\overline{\partial}^{-1}\overline{a}_1 + \partial^{-1} a_1 = \overline{\partial}^{-1}\overline{a}_2 + \partial^{-1} a_2$ in Ω. □

In Vekua [28], we have the following lemma.

Lemma 4.2. *Suppose that $f \in L^1(\mathbb{R}^2)$. Then we have*

$$\overline{\partial}(\overline{\partial}^{-1} f) = f \quad \text{and} \quad \partial(\partial^{-1} f) = f.$$

Moreover, if $f \in C^1(\mathbb{R}^2)$, then we obtain the following result:

$$\overline{\partial}\partial(\overline{\partial}^{-1} f) = \partial f \quad \text{and} \quad \partial\overline{\partial}(\partial^{-1} f) = \overline{\partial} f.$$

With the result of Theorem 2.2, we have

$$\overline{\partial}^{-1}\overline{a}_1 + \partial^{-1} a_1 = \overline{\partial}^{-1}\overline{a}_2 + \partial^{-1} a_2 \text{ in } \Omega. \tag{64}$$

Let the operator $\overline{\partial}\partial$ acts on both sides of (64). Therefore, by lemma 4.2, we obtain

$$\partial\overline{a}_1 + \overline{\partial} a_1 = \partial\overline{a}_2 + \overline{\partial} a_2 \text{ in } \Omega.$$

For $j = 1, 2$, we have the equalities

$$\frac{1}{2}(\partial\overline{a}_j + \overline{\partial} a_j) = \frac{1}{4} rot \overrightarrow{A}_j \text{ in } \Omega.$$

Then, it can be deduced that

$$\frac{1}{4} rot \overrightarrow{A}_1 = \frac{1}{4} rot \overrightarrow{A}_2 \text{ in } \Omega.$$

According to the above observation, Theorem 1.1 is a direct consequence of Theorem 2.2.

5. **Reconstruction of the magnetic field and the electric potential.** To reconstruct the magnetic field and the electric potential, Salo used a similar method as in Nachman [14]. But, here we applied the new result from Bukhgeim [3] to reconstruct both of them.

We have mentioned the following operator in the above section.

$$P := \tilde{D} - B$$

where $\tilde{D}$ and B are denoted by

$$\tilde{D} = \begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} \quad ; \quad B = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$$

where a, b are complex functions. Suppose that $\varphi \in C^2(\overline{\Omega})$ is a real-valued function. Moreover, there is a point $x_0 \in \Omega$ such that $\nabla\varphi(x_0) = 0$ and $det(\partial_j\partial_k\varphi(x_0)) \neq 0$. Define

$$\Phi = \begin{pmatrix} \psi & 0 \\ 0 & \overline{\psi} \end{pmatrix}$$

with $\overline{\partial}\psi = 0$ and $2Im\psi = \varphi$. For $\tau > 0$, considering the Riemann-Hilbert problem

$$\begin{cases} Pu = 0 & \text{in } \Omega; \\ Re(e^{-\tau\Phi}u) = f & \text{on } \partial\Omega. \end{cases} \tag{65}$$

We recall, without proof, the following result proved in [3].

Theorem 5.1. *(Bukhgeim, 2008) If $a,\ b \in C^1(\Omega)$, then there exists $\tau_0 > 0$ such that for all $\tau \geq \tau_0$, we have a solution of the Riemann-Hilbert problem*

$$\begin{cases} Pu = 0 & \textit{in } \Omega; \\ Re(e^{-\tau\Phi}u) = f & \textit{on } \partial\Omega. \end{cases} \tag{66}$$

Moreover, the solution has the form

$$u = e^{\tau\Phi}(v + \omega),$$

where

$$\tilde{D}v = 0 \textit{ in } \Omega,\ \ Rev|_{\partial\Omega} = f \textit{ and } Re\omega|_{\partial\Omega} = 0.$$

Since v satisfies

$$\begin{cases} \tilde{D}v = 0 & \text{in } \Omega; \\ Rev = f & \text{on } \partial\Omega, \end{cases} \tag{67}$$

there exists an explicit solution for (67). More precisely, v can be expressed as

$$v_1(z) = \frac{1}{2\pi i}\int_{\partial\Omega} f_1(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} + ic_1,$$
$$v_2(z) = \overline{\frac{1}{2\pi i}\int_{\partial\Omega} f_2(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta}} + ic_2,$$

where $c_1,\ c_2 \in \mathbb{R}$. See [28] for details.

We follow the lines parallel to [3]. Considering the first order system $P = \tilde{D} - Q$ with

$$\tilde{D} = \begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & e^{Ta}q \\ e^{-Ta} & 0 \end{pmatrix},$$

which is discussed in Section 3. We still consider the problem

$$\begin{cases} Pu = \tilde{D}u - Qu = 0 & \text{in } \Omega; \\ Re(e^{-\tau\Phi}u) = f & \text{on } \partial\Omega \end{cases} \tag{68}$$

with the solution u of the form

$$u = e^{\tau\Phi}(v + \omega).$$

Here we give a special case. Let $f = e_1 = (1, 0)$. We have

$$v|_{\partial\Omega} = e_1 + ic \text{ and } \omega|_{\partial\Omega} = ig(x, \tau),$$

where c and $g(x, \tau) \in \mathbb{R}^2$. Considering the image part of $e^{-\tau\Phi}u$, we obtain

$$Im(e^{-\tau\Phi}u)|_{\partial\Omega} = Im(e_1 + ic + \omega)|_{\partial\Omega} = c + g(x, \tau).$$

Given $\mu = e^{-\tau\Phi}e_2$. Denote $\nu = (\nu_1 + i\nu_2)$ be the unit outer normal to $\partial\Omega$. By Green's formula, since $\overline{\partial}\psi = 0$, we obtain

$$\begin{aligned}\int_\Omega < \tilde{D}u, \mu > dx &= \int_\Omega \left\langle \begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} 0 \\ e^{-\tau\overline{\psi}} \end{pmatrix} \right\rangle dx \\ &= \int_\Omega (\partial u_2) e^{-\tau\overline{\psi}} dx \\ &= -\int_\Omega u_2 \partial e^{-\tau\overline{\psi}} dx + \frac{1}{2}\int_{\partial\Omega} \overline{\nu} u_2 e^{-\tau\overline{\psi}} ds \\ &= \frac{1}{2}\int_{\partial\Omega} \overline{\nu} u_2 e^{-\tau\overline{\psi}} ds.\end{aligned}$$

Since $e^{-\tau\Phi}u = (e^{-\tau\psi}u_1, e^{-\tau\overline{\psi}}u_2)$, we obtain $e^{-\tau\overline{\psi}}u_2 = v_2 + \omega_2$. Then

$$\begin{aligned}\int_\Omega < \tilde{D}u, \mu > dx &= \frac{1}{2}\int_{\partial\Omega} \overline{\nu}(v_2 + \omega_2) ds \\ &= \frac{1}{2}\int_{\partial\Omega} \overline{\nu}(ic_2 + ig_2) ds \\ &= \frac{1}{2} i \int_{\partial\Omega} \overline{\nu} g_2 ds.\end{aligned}$$

As in [3], $e^{-Ta(x_0)}$ can be approached by

$$\tau \int_\Omega < Qu, \mu > dx \to 2\pi e^{-Ta(x_0)} \quad \text{as } \tau \to \infty$$

if $\phi(x) = \frac{1}{2}[(x_1 - x_{01})^2 - (x_2 - x_{02})^2]$, where $x_0 = (x_{01}, x_{02}),\ x = (x_1, x_2)$. Because of

$$\int_\Omega < \tilde{D}u, \mu > dx = \int_\Omega < Qu, \mu > dx,$$

we can deduce that

$$e^{-Ta(x_0)} = \frac{-1}{4\pi i} \lim_{\tau\to\infty} \tau \int_{\partial\Omega} \overline{\nu} g_2 ds. \tag{69}$$

Since $Ta(z) = (\overline{\partial}^{-1}\overline{a} + \partial^{-1}a)(z)$, it follows that

$$(\overline{\partial}^{-1}\overline{a} + \partial^{-1}a)(x_0) = \ln\left(\frac{1}{4\pi i} \lim_{\tau\to\infty} \tau \int_{\partial\Omega} \overline{\nu} g_2 ds\right).$$

Using the fact that $\overline{\partial}\partial(\overline{\partial}^{-1}f) = \partial f$, we have

$$(\partial\overline{a} + \overline{\partial}a)(x_0) = \overline{\partial}\partial\left(\ln\left(\frac{1}{4\pi i} \lim_{\tau\to\infty} \tau \int_{\partial\Omega} \overline{\nu} g_2 ds\right)\right).$$

Because $\partial\overline{a} + \overline{\partial}a = \frac{1}{2} rot\overrightarrow{A}$ in Ω, we get

$$(rot\overrightarrow{A})(x_0) = \overline{\partial}\partial\left(\ln\left(\frac{1}{4\pi i} \lim_{\tau\to\infty} \tau \int_{\partial\Omega} \nu g_2 ds\right)\right).$$

Therefore, we get a formula to compute the magnetic field $rot\overrightarrow{A}$ in Ω.

To give a formula to the electric potential q. Considering $f = e_2 = (0, 1)$ and $\eta = e^{-\tau\Phi}e_1$. Following similar process, we obtain

$$\int_\Omega < \tilde{D}u, \eta > dx = \frac{1}{2} i \int_{\partial\Omega} \nu g_1 ds.$$

Hence, $e^{Ta(x_0)}q$ can be written by

$$e^{Ta(x_0)}q = \frac{-1}{4\pi i} \lim_{\tau\to\infty} \tau \int_{\partial\Omega} \nu g_1 ds. \tag{70}$$

From (69) and (70) , q can be formulated by

$$q(x_0) = \frac{-1}{(4\pi)^2} \left(\lim_{\tau\to\infty} \tau \int_{\partial\Omega} \overline{\nu} g_2 ds \right) \left(\lim_{\tau\to\infty} \tau \int_{\partial\Omega} \nu g_1 ds \right).$$

REFERENCES

[1] Kari Astala and Lassi Päivärinta, *Calderón's inverse conductivity problem in the plane*, Annals of Math., **163** (2006), 265–299.

[2] Russell M. Brown, *Recovering the conductivity at the boundary from the Dirichlet to Neumann map: a pointwise result*, J. Inv. Ill-Posed Problems, **9(6)** (2001), 567-574.

[3] A. L. Bukhgeim, *Recovering a potential from Cauchy data in the two-dimensional case*, J. Inv. Ill-Posed Problems, **16** (2008), 19–33.

[4] R. Beals and R. R. Coifman, *Multidimensional inverse scatterings and nonlinear partial differential equations*, In F. Treves, editor Pseudodifferential operators and applications, proceedings of symposia in pure mathematics, Amer. Math. Soc., **43** (1985), 45–70.

[5] R. Beals and R. R. Coifman, *The spectral problem for the Davey-Stewartson and Ishimori hierarchies*, In Nonlinear evolution equations: Integrability and spectral methods, Manchester University Press, (1988), 15–23.

[6] Russell M. Brown and Mikko Salo, *Identifiability at the boundary for first-order terms*, Applicable Analysis, **85** (2006), 735-749.

[7] Russell M. Brown and Gunther Uhlmann, *Uniquenes in the inverse conductivity problem for nonsmooth conductivities in two dimensions*, Comm. in Partial Differential Equations, **22** (1997), 1009-1027.

[8] A. P. Calderón, *On an inverse boundary value problem*, in Seminar on Numerical Analysis and its Applications to Continuum Physics, Rí o de Janeiro, (1980).

[9] Jin Cheng, Gen Nakamura and Erkki Somersalo, *Uniquenss of identifying the convection term*, Comm. Korean Math. Soc., **16** (2001), 405-413.

[10] Jin Cheng and Masahiro Yamamoto, *Determination of two convection coefficients from Dirichlet to Neumann map in the two-dimensional case*, SIAM J. Math. Anal., **35(6)** (2004), 1371-1393.

[11] G. Eskin and J. Ralston, *Inverse scattering problem for the Schrödinger equation with magnetic potential at a fixed energy*, Comm. Math. Phys., **173** (1995), 199-224.

[12] Hyeonbae Kang and Gunther Uhlmann, *Inverse problems for the Pauli Hamiltonian in two dimensions*, The Journal of Fourier Analysis and Applications, **10** (2004), 201-215.

[13] R. Kohn and M. Vogelius, *Determining conductivity by boundary measurements*, Comm. Pure Appl. Math., **37** (1984), 289-298.

[14] Adrian I. Nachman, *Reconstructions from boundary measurements*, Annals of Math., **128** (1988), 531-576.

[15] Adrian I. Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem*, Annals of Math., **142** (1995), 71-96.

[16] Gen Nakamura, Ziqi Sun and Gunther Uhlmann, *Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field*, Math. Ann, **303** (1995), 377-388.

[17] Louis Nirenberg and Homer F. Walker, *The null spaces of elliptic partial differential operators in $\mathbb{R}^n$*, Journal of Math. Analysis and Applications, **42** (1973), 271-301.

[18] Mikko Salo, *Semiclassical pseudodifferential calculus and the reconstuction of a magnetic field*, Comm. in PDE, **31** (2006), 1639-1666.

[19] Mikko Salo, *Inverse problems for nonsmooth first order perturbations of the laplacian*, Dissertationes, **139**.

[20] Ziqu Sun, *An inverse boundary value problem for Schrödinger operators with vector potentials*, Trans. of AMS, **338** (1993), 953-971.

[21] Ziqu Sun, *An inverse boundary value problem for Schrödinger operators with vector potentials in two dimensions*, Comm. in PDE, **18** (1993), 83-124.

[22] John Sylveser, *An anisotropic inverse boundary value problem*, Comm. on Pure and Applied Math., **43** (1990), 201-232.

[23] John Sylvester and Gunther Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Annals of Math., **125** (1987), 153-169.

[24] John Sylvester and Gunther Uhlmann, *A uniqueness theorem for an inverse boundary value problem in electrical prospection*, Comm. Pure and Applied Math., **39** (1986), 91-112.

[25] Carlos F. Tolmasky, *Exponentially growing solutions for nonsmooth first-order perturbations of the laplacian*, SIAM J. Math. Anal., **29(1)** (1998), 116-133.

[26] Gunther Uhlmann, *Developments in inverse problems since Calderón's foundational paper*, Chapter 19 in "Harmonic Analysis and Partial Differential Equations," University of Chicago Press, (1999), 295-345, edited by M. Christ, C. Kenig and C. Sadosky.

[27] Gunther Uhlmann, *Electrical impedance tomography and Calderón's problem*, Inverse Problems.

[28] I. N. Vekua, *Generalized analytic functions*, 2nd ed. Pergamon Press, London, (1962).

Received December 2009; revised November 2010.

E-mail address: rylai@u.washington.edu