Lecture 4: Quick review from previous lecture

- We have discussed two type of elementary row operators:
  - type 1 - adding/subtracting one row to another row;
  - type 2 - permutation (pivoting)
- We learned how to use “LU factorization” to solve a linear system $A\mathbf{x} = \mathbf{b}$.
- $A$ is nonsingular $\iff A$ has a permuted $LU$ factorization: $PA = LU$

- Today we will systematically build $L$, $U$ and $P$ for a nonsingular matrix.
- We also will discuss the inverse of a matrix.

- Lecture will be recorded -

- The first homework will due this Friday (9/18) by 6pm.
  - HW2 is posted.
Example. Find LU factorization for the matrix

\[ A = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 3 & -3 \\ -2 & -6 & -2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix} \]

Performing 2\textsuperscript{nd} type of elementary row operation (permuting rows) indeed can give: \( P A \) is regular (has all nonzero pivot). Then we can find its \( LU \) factorization, namely, \( P A = LU \).
The permuted $LU$ factorization can be used to solve linear systems $Ax = b$.

\[ PAx = \begin{bmatrix} \text{Pb} \end{bmatrix} \quad \text{(multiply P on both sides of } Ax = b) \]

1. Solve $Ly = \begin{bmatrix} \text{Pb} \end{bmatrix}$ (solve for "y" by forward substitution)

2. Then solve $Ux = y$ (solve for "x" for back substitution)

Example. Let $A$ be the matrix from the previous example. Solve $Ax = b$, where

\[ b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \]

Since we already knew that $PA = LU$,

\[ PAx = Pb \Rightarrow L(Ux) = \begin{bmatrix} \text{Pb} \end{bmatrix} \]

1. Solve $Ly = \begin{bmatrix} \text{Pb} \end{bmatrix}$

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
2 & 0 & 1 & 0 \\
-2 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{pmatrix}
= \begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}
\]

By forward substitution,

\begin{align*}
\text{1.} & \quad y_1 = 1 \\
\text{2.} & \quad y_1 + y_2 = 4 \implies y_2 = 3 \\
\text{3.} & \quad \text{...} \\
\text{4.} & \quad \text{...}
\end{align*}
Fact: If $A$ is square and nonsingular, then $Ax = b$ has a unique solution $x$ for any right hand side $b$. 

Then $y = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix}$.

2. Solve $Ux = y$.

By back substitution,

$4. x_4 = 1$

$3. x_3 - 7(1) = 0 \Rightarrow x_3 = 7$

$2. -x_2 + 0 + (1) = 3 \Rightarrow x_2 = -2$

$1. \ldots \Rightarrow x_1 = -2$.
1.5 Matrix Inverse

The inverse of a matrix is analogous to $a^{-1} = \frac{1}{a}$ of a scalar $a \neq 0$. Thus, for [5] 1 by 1 matrix, it has inverse $[\frac{1}{5}]$. Then

$[5][\frac{1}{5}] = [1]$.

- $A$ is a square matrix. Then we call an $n \times n$ matrix $X$ the inverse of $A$ if it satisfies

$$AX = I_n = XA.$$  

We then denote such matrix $X$(inverse of $A$) by $A^{-1}$.

Note that ”Not every matrix has an inverse !!!”

**Example.** The inverse of the matrix

$$E = \begin{pmatrix}
1 & 2 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \text{elementary matrix} \quad \text{"add 2 \circ\ circ to 1"}.$$  

is simply the matrix

$$E^{-1} = \begin{pmatrix}
1 & -2 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.$$  

Check

$$E^{-1}E = I_3.$$  

$$EE^{-1} = I_3.$$
Example. A permutation matrix

\[
P = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

"permute ① ② ③".

Thus, \[P^{-1} = P\]

- In general, however, finding \(A^{-1}\) will not be so easy. We will see a systematic method for doing so in the next class, known as Gauss-Jordan elimination.

- In the following, we will discuss 3 key facts:

Fact 1. If the inverse of a matrix exists, then this inverse matrix is unique. In other words, if \(B\) and \(C\) are both inverse of \(A\), then \(B = C\).

Proof: Since \(B\), \(C\) are inverse of \(A\), by definition

\[
BA = AB = I, \\
CA = AC = I
\]

associative

\[
B = BI = B(AC) = (BA)C = IC = C.
\]

Fact 2. The inverse of the inverse is the original matrix. More precisely, \((A^{-1})^{-1} = A\).

Proof. This is an immediate consequence of the defining property of \(A^{-1}\).

\[
A A^{-1} = I = A^t A, \quad A^{-1} = (A^t)^{-1}.
\]
• Continue the 3 key facts:

**Fact 3.** If $A$ and $B$ are two invertible $n$-by-$n$ matrices, then their product $AB$ is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$. (sometimes called "shoes and socks" theorem)

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**Remark:** In general,

$$(A_1A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}.$$
(9/14) Poll Question 1: If $A, B, C$ are invertible $n \times n$ matrices, then

A) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$  
B) $(ABC)^{-1} = A^{-1}B^{-1}C^{-1}$  
C) $(ABC)^{-1} = C^{-1} + B^{-1} + A^{-1}$  

* You should be able to see the pop up Zoom question. Answer the question by clicking the corresponding answer and then submit.

Caution: after clicking submit, you will not be able to resubmit your answer!