Lecture 9: Quick review from previous lecture

• If

\[
A \rightarrow U = \begin{pmatrix}
    u_{11} & \cdots & \cdots & \cdots \\
    0 & u_{22} & \cdots & \cdots \\
    \vdots & \ddots & \ddots & \ddots \\
    0 & \cdots & 0 & u_{nn}
\end{pmatrix},
\]

then

\[
\det(A) = (-1)^k \prod_{i=1}^{n} u_{ii} = c \text{if } u_{11} \cdots u_{nn}
\]

where \(k\) denotes the number of row permutations we performed to bring \(A\) into upper triangular form.

• Let \(A\) is \(n \times n\) matrix.

\[
\det(cA) = c^n \det(A)
\]

• If \(A\) is invertible, then

\[
\det(A^{-1}) = \frac{1}{\det(A)}
\]

\[
\det(A) \neq 0 \iff A \text{ is nonsingular (invertible)}
\]

Today we will discuss

• Sec. 2.1 vector space

- Lecture will be recorded -

• More detailed instructions for Exam 1 on 10/14 will be announced soon.

• The exam will be a closed book exam and everyone needs to open Camera.
2.1 Real Vector Spaces

**Definition:** A **vector space** is a set $V$ equipped with two operations:

1. **(Addition)** If $v, w \in V$, then $v + w \in V$.
2. **(Scalar Multiplication)** Multiplying a vector $v \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $cv \in V$.

For all $u, v, w \in V$ and all scalars $c, d \in \mathbb{R}$:

- (a) **Commutativity of Addition:** $v + w = w + v$.
- (b) **Associativity of Addition:** $u + (v + w) = (u + v) + w$.
- (c) **Additive Identity:** There is a **zero element** $0 \in V$ satisfying $v + 0 = v = 0 + v$.
- (d) **Additive Inverse:** For each $v \in V$ there is an element $-v \in V$ such that $v + (-v) = 0 = (-v) + v$.
- (e) **Distributivity:** $(c + d) v = (cv) + (dv)$, and $c(v + w) = (cv) + (cw)$.
- (f) **Associativity of Scalar Multiplication:** $c(dv) = (cd)v$.
- (g) **Unit for Scalar Multiplication:** the scalar $1 \in \mathbb{R}$ satisfies $1 v = v$.

**Example 1.** $\mathbb{R}^n$ is a **vector space** with addition and scalar multiplication defined by

$$
\begin{align*}
\mathbf{u} + \mathbf{v} &= (a_1 + b_1, \ldots, a_n + b_n)^T, \\
\mathbf{c} \mathbf{u} &= (ca_1, \ldots, ca_n)^T
\end{align*}
$$

where $\mathbf{u} = (a_1, \ldots, a_n)^T$ and $\mathbf{v} = (b_1, \ldots, b_n)^T$.

Check (1), (2), (ok)

(a) $\mathbf{u} + \mathbf{v} = (a_1 + b_1, \ldots, a_n + b_n)^T = (b_1 + a_1, \ldots, b_n + a_n)^T$

(b) **true**

(c) **true**

(d) **true**

(g).
Example 2. The set of all \( m \times n \) matrices with entries from \( \mathbb{R} \) is a vector space, which we denote by \( M_{m \times n}(\mathbb{R}) \), with the following operations of **matrix addition** and **scalar multiplication**: For \( A, B \in M_{m \times n}(\mathbb{R}) \) and \( c \in \mathbb{R} \),

\[
(A + B)_{ij} = A_{ij} + B_{ij}, \quad (cA)_{ij} = cA_{ij}
\]

For instance, \( M_{2 \times 3}(\mathbb{R}) \) is a vector space.

Example 3. Consider the space

\[
P^{(n)} = \{ p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \}
\]

consisting of all real polynomials of degree \( \leq n \).

We have a way to define addition of two polynomials of degree \( \leq n \):

If \( p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) and \( q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0 \), then

\[
(p + q)(x) = (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \cdots + (a_1 + b_1) x + (a_0 + b_0)
\]

\[
cp(x) = ca_n x^n + ca_{n-1} x^{n-1} + \cdots + ca_1 x + ca_0
\]

Check if \( \text{III.} \times \), \( \text{OK.} \):

(a) \( A + B = B + A \).

(b) \( \ldots \).

(c) Zero element \( 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) in \( M_{2 \times 3}(\mathbb{R}) \). \( O + A = A = A + O \).

(d) \( A + (-A) = O. \) \( \ldots \)

(g) \( 0 \) in \( P^{(1)} \).

If \( \text{check a, if} \ (a) \), \( \text{ok, if} \ (c) \) Zero element \( 0 \) in \( P^{(n)} \). \( 0 + p = p = 0 + p \). \( \text{fine} \) (c) \( a_n = 0, \ldots, a_0 = 0 \) zero polynomial.

\( P^{(n)} \) is a vector space.
Example 4. Consider the set $S$ of polynomials of degree equal to $n$ with same addition and scalar multiplication as in Example 3. Then $S$ is not a vector space.

\[
S = \{ a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0 \mid a_n \neq 0 \}
\]

$\begin{align*}
p &= x^2, \\
qu &= -x^2 \\
p + q &= 0 \notin S \quad (0 \text{ is NOT in } S)
\end{align*}$

Example 5. We consider the set

\[
S = \{ f \in \mathcal{F}([a, b]) : f(a) = 1 \}
\]

and we define add two functions in $S$ and multiply function in $S$ by a scalar $c$ as follows:

\[
(f + g)(x) = f(x) + g(x), \quad (c \cdot f)(x) = c \cdot f(x)
\]

Is $S$ a vector space? Here $\mathcal{F}([a, b])$ is the collection of all functions $f$ defined on an interval $[a, b]$.

Check (1): $f, g \in S, \quad f(a) = 1, \quad g(a) = 1$. 

\[
(f + g)(a) = f(a) + g(a) = 2.
\]

Thus $f + g \notin S$. Contradicate (1).

Q: Is $S' = \{ f \in \mathcal{F}([a, b]) : f(a) = 0 \}$ with same addition and scalar multiplication as above a vector space? (Yes).
Example 6. Let $S$ be a set $\{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. We define the addition and scalar multiplication by

\[(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2), \quad c(a_1, a_2) = (ca_1, ca_2)\]

Is $S$ a vector space?

Check:

1. $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$
2. $c(a_1, a_2) = (ca_1, ca_2)$

Example 7. We let $V$ be the upper right quadrant of $\mathbb{R}^2$, i.e.

\[V = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}\]

We define addition of vectors and scalar multiplication by:

\[(x, y) \oplus (w, z) = (xw, yz), \quad c \odot (x, y) = (x^c, y^c)\]

* Here we use these notations $\oplus, \odot$ to distinguish it from ordinary ones.

Is $V$ a vector space?

Yes:

1. $(x, y) \oplus (w, z) = (xw, yz)$ in $V$
2. $c \odot (x, y) = (x^c, y^c)$ in $V$
3. $(x, y) \oplus (w, z) = (xw, yz) = (w, z) \oplus (x, y)$.
4. $\text{the}$
5. zero element $(a, b) = (1, 1)$.
6. $(a, b) \oplus (x, y) = (x, y)$.
7. $a = 1, \ b = 1$.
8. $(x, y) \in V$.
9. $(x, y) \oplus \left(\frac{1}{x}, \frac{1}{y}\right) = (1, 1)$.

For (e) and (g) true.