Today we will learn how to solve a **linear system**.

For example:

\[
\begin{align*}
5x + 7y + 3z &= 2 \\
2x + y + 6z &= 1 \\
x + 10y + 3z &= 5
\end{align*}
\]

(3 equations with 3 unknowns \(x, y, z\)), or

\[
\begin{align*}
w + 5x + 7y + 3z &= 2 \\
2w + 2x + y + 6z &= 1 \\
3w + x + 10y + 3z &= 5 \\
2w + 9x + 4y + 2z &= 7
\end{align*}
\]

(4 equations with 4 unknowns \(w, x, y, z\)).

Given such a system of equations, we want to find the variables \(x, y, z, \ldots\) that satisfy all equations simultaneously.

We will learn **Gaussian Elimination**, that is to reduce the original system to a much simpler system that still has the same solution.
Example: Find solution of linear system:

\[
\begin{align*}
  x - 2y + z &= 3 \quad - \quad 1 \\
  2x - y - 2z &= 6 \quad - \quad 2 \\
  3x - 7y + 4z &= 10 \quad - \quad 3
\end{align*}
\]

\text{Eqn. 2} - 2 \text{ Eqn. 1} \rightarrow \begin{align*}
  2x - y - 2z &= 6 \\
  -) 2x - 4y + 2z &= 6 \\
  \underline{3y - 4z = 0} \quad \text{New 2}
\end{align*}

New system:
\[
\begin{align*}
  x - 2y + z &= 3 \\
  3y - 4z &= 0 \\
  3x - 7y + 4z &= 10
\end{align*}
\]

\text{New system has the same solution as original system.}

\text{3 - 3(1)} \rightarrow \begin{align*}
  x - 2y + z &= 3 \\
  3y - 4z &= 0 \\
  -y + z &= 1
\end{align*}

Now we have eliminated \(x\) from \(2, 3\).  \(3 - 8 + 3 = -2\)

\text{3 + \frac{1}{3}(2)} \rightarrow \begin{align*}
  x - 2y + z &= 3 \\
  3y - 4z &= 0 \\
  -\frac{1}{3}z &= 1
\end{align*}

\[\frac{1}{3}z = y - \frac{4}{3}z = 0 \quad +) -y + z = 1 \quad \frac{1}{3}z = 1\]

\[\frac{1}{3}z = 1 \quad \Rightarrow \quad z = -3 \]

Plug in \(z = -3, \quad y = \frac{4}{3}z = -4, \quad x = 3 + 2y - z\)

\[x = -2, \quad y = -4, \quad z = -3\]

Remark (RK): "This process is called "Gaussian Elimination"."
Use $1$, to eliminate 1st variable from $2$...$n$

Use $2$... $2$nd $3$... $n$

Gaussian Elimination get an upper triangular system.

We solve it from bottom up, called "back substitution".
1.2 Matrices and Vectors and Basic Operations

• A **matrix** is simply a rectangle array of numbers, such as,

\[
\begin{bmatrix}
1 & 0.7 & 10 \\
\pi & 6 & 0
\end{bmatrix}_{2 \times 3},
\begin{bmatrix}
\cos(1) & 1 \\
4 & 6 \\
-10 & e^2
\end{bmatrix}_{3 \times 2}
\]

The 1st matrix above is a $2 \times 3$ matrix and 2nd matrix above is a $3 \times 2$ matrix.

• Generally, an $m \times n$ matrix $A$ is a two-dimensional array of $m \cdot n$ numbers:

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

where $m$ is the number of rows and $n$ is the number of columns. The element $a_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$, is called the **entry** of $A$.

• A column vector is a matrix where $n = 1$:

\[
v = \begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_m
\end{pmatrix}
\]

• A row vector is a matrix where $m = 1$:

\[
w = (w_1 w_2 \cdots w_n)
\]
Three basic operations:

1. Matrix addition:

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
+ \begin{pmatrix}
  b_{11} & b_{12} & \cdots & b_{1n} \\
  b_{21} & b_{22} & \cdots & b_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{m1} & b_{m2} & \cdots & b_{mn}
\end{pmatrix}
= \begin{pmatrix}
  a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\
  a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn}
\end{pmatrix}
\]

2. Scalar multiplication: If \( c \) is a number, we can multiply a matrix by \( c \):

\[
c \times \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
= \begin{pmatrix}
  c \cdot a_{11} & c \cdot a_{12} & \cdots & c \cdot a_{1n} \\
  c \cdot a_{21} & c \cdot a_{22} & \cdots & c \cdot a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  c \cdot a_{m1} & c \cdot a_{m2} & \cdots & c \cdot a_{mn}
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 \\
  4
\end{pmatrix}
\begin{pmatrix}
  3 \\
  6
\end{pmatrix}
= \begin{pmatrix}
  7 \\
  28
\end{pmatrix}
\]

3. Matrix multiplication:

\[
\begin{pmatrix}
  v_1 & \cdots & v_p
\end{pmatrix}
\begin{pmatrix}
  w_1 \\
  \vdots \\
  w_p
\end{pmatrix}
= v_1 w_1 + v_2 w_2 + \cdots + v_p w_p
\]
Generally, if $A = (a_{ij})$ is $m \times n$ matrix and $B = (b_{ij})$ is $n \times p$ matrix, then their product $C = AB$ is $m \times p$ matrix and has entries:

$$c_{ij} = (i^{th} \text{ row of } A) \times (j^{th} \text{ column of } B)$$

Remark:

- Matrix multiplication is associative: $(AB)C = A(BC)$
- Not commutative: in general, $AB \neq BA$. 

Remark:
Example: \( A = (1, 2, 3) \) and 

\[
B = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.
\]

Compute \( AB \) and \( BA \).

\[
AB = \begin{pmatrix} 1 & 2 & 3 \\ \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \left( 4 + 10 + 18 \right) = (32).
\]

\[
BA = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{pmatrix}_{3 \times 3}
\]

Vectors and matrices provide a convenient notation for linear systems.

- For example, the linear system

\[
\begin{align*}
5x - 4y + 3z &= 19 \\
4x - 5y + 3z &= 17 \\
x - y - 2z &= 4
\end{align*}
\]

is equivalent to the equation:

\[
\begin{pmatrix} 5 & -4 & 3 \\ 4 & -5 & 3 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 19 \\ 17 \\ 4 \end{pmatrix}
\]

- In more compact notation, we can write:

\[
Ax = b
\]

where

\[
A = \begin{pmatrix} 5 & -4 & 3 \\ 4 & -5 & 3 \\ 1 & -1 & -2 \end{pmatrix}, \quad x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad b = \begin{pmatrix} 19 \\ 17 \\ 4 \end{pmatrix}
\]
Some special matrices we will see and utilize many times in this course.

- **The \( n \)-by-\( n \) identity matrix**, typically denoted \( I \) or \( I_n \), defined by:

\[
I = I_n = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}_{n \times n}
\]

In other words, \( I \) has 1’s on the main diagonal, and the off-diagonal elements are 0. It’s easy to check that

\[I_n A = A \quad \text{and} \quad B I_n = B\]

for any matrix \( A \) with \( n \) rows and any matrix \( B \) with \( n \) columns.

- **The \( m \)-by-\( n \) zero matrix**, typically denoted \( O \) or \( O_{m \times n} \), which has all zero entries. It’s easy to check that

\[O_{m \times n} A = O_{m \times k}\]

for any \( n \)-by-\( k \) matrix \( A \), and

\[B O_{m \times n} = O_{k \times n}\]

for any \( k \)-by-\( m \) matrix \( B \).

- Some useful notations:

  - if \( a_1, a_2, \ldots, a_n \) are \( n \) numbers, we will denote by \( \text{diag}(a_1, \ldots, a_n) \) the following \( n \)-by-\( n \) matrix:

\[
\text{diag}(a_1, \ldots, a_n) = \begin{pmatrix}
a_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & a_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & a_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{n-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & a_n
\end{pmatrix}
\]
In other words, diag\((a_1, \ldots, a_n)\) has \(a_1, \ldots, a_n\) on the main diagonal, and the off-diagonal elements are 0.

- In this notation, \(I_n = \text{diag}(1, \ldots, 1)\) \(n\) times

§ The **augmented matrix** for a linear system appends the right hand side as an extra column to the coefficient matrix.

- For example, the augmented matrix for the linear system

\[
\begin{align*}
 x + 2y + 2z &= 2 \\
 2x + 6y &= 1 \\
 4x + 4z &= 0
\end{align*}
\]

is the 3-by-4 matrix:

\[
\begin{pmatrix}
1 & 2 & 2 & 2 \\
2 & 6 & 0 & 1 \\
4 & 0 & 4 & 0
\end{pmatrix}
\]

- For clarity, this may also be written like this:

\[
\begin{pmatrix}
1 & 2 & 2 & 2 \\
2 & 6 & 0 & 1 \\
4 & 0 & 4 & 0
\end{pmatrix}
\]

- Gaussian elimination can be expressed entirely in terms of the augmented matrix.

- Also, the operations of Gaussian elimination can be used to update the augmented matrix.
Example:

\[
\begin{pmatrix}
1 & 2 & 2 & | & 2 \\
2 & 6 & 0 & | & 1 \\
4 & 0 & 4 & | & 0
\end{pmatrix}
\begin{array}{l}
\text{is identical to} \\
\begin{cases}
x + 2y + 2z = 2 & - \text{1} \\
x + 2y + 2z = 2 & - \text{1} \\
2x + 6y = 1 & - \text{2} \\
4x + 4z = 0 & - \text{3}
\end{cases}
\end{array}
\]

\[
\begin{pmatrix}
1 & 2 & 2 & | & 2 \\
0 & 2 & -4 & | & -3 \\
0 & -8 & -4 & | & -8
\end{pmatrix}
\begin{array}{c}
\text{eliminate } "x" \\
\text{in 1,2}
\end{array}
\begin{array}{l}
x + 2y + 2z = 2 \\
-2y + 4z = -3
\end{array}
\]

\[
\begin{pmatrix}
1 & 2 & 2 & | & 2 \\
0 & 2 & -4 & | & -3 \\
0 & 0 & -8 & | & -8
\end{pmatrix}
\begin{array}{c}
\text{eliminate } "y" \text{ in 2}
\end{array}
\begin{array}{l}
x + 2y + 2z = 2 \\
2y - 4z = -3 \\
-8y - 4z = -8
\end{array}
\]

Using "back - substitution",

\[
\begin{align*}
z &= 1, & \text{2} & \quad 2y = -3 + 4z = -3 + 4 = 1, \quad y = \frac{1}{2}.
\end{align*}
\]

Plug \( z = 1, \ y = \frac{1}{2} \) into \( 1 \), \( x = 2 - 2y - 2z = 1 = 2 - 1 - 2 = -1 \)

Remark:

- Adding/subtracting a multiple of one row to/from another row is called an elementary row operation.
- Each elementary row operation is associated with an elementary matrix, defined by applying the elementary row operation to the identity matrix.
Example. The elementary matrix associated with adding 3 times the 3rd row to the 1st row is:

\[
E = \begin{pmatrix}
1 & 0 & 3 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

• Multiplying a matrix \( A \) on the left by an elementary matrix \( E \) performs the associated row operation on \( A \). For example, check that:

\[
\begin{pmatrix}
1 & 0 & 3 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \times \begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{pmatrix} = \begin{pmatrix}
a + 3g & b + 3h & c + 3i \\
d & e & f \\
g & h & i
\end{pmatrix}
\]