Chapter 1. Linear Algebraic Systems
Lecture 1: 1.1 The Solution of Linear Systems

- Today we will learn how to solve a **linear system**.

  For example:

$$\begin{aligned} 5x + 7y + 3z &= 2 \\ 2x + y + 6z &= 1 \\ x + 10y + 3z &= 5 \end{aligned}$$

  (3 equations with 3 unknowns x, y, z), or

$$\begin{aligned} w + 5x + 7y + 3z &= 2 \\ 2w + 2x + y + 6z &= 1 \\ 3w + x + 10y + 3z &= 5 \\ 2w + 9x + 4y + 2z &= 7 \end{aligned}$$

  (4 equations with 4 unknowns w, x, y, z).

- Given such a system of equations, we want to find the variables $x, y, z, \ldots$ that satisfy all equations simultaneously.

- We will learn **Gaussian Elimination**, that is to reduce the original system to a much simpler system that still has the same solution.

Example: Find solution of linear system:

$$x - 2y + z = 3 \quad — ①$$
$$2x - y - 2z = 6 \quad — ②$$
$$3x - 7y + 4z = 10 \quad — ③$$

$\xrightarrow{\text{Equ. ② } - 2 \text{ Equ. ①}}$

$$\begin{array}{r} 2x - y - 2z = 6 \\ -)\ 2x - 4y + 2z = 6 \\ \hline 3y - 4z = 0 \end{array} \quad \text{New ②}$$

New system:

$$\begin{cases} x - 2y + z = 3 \\ 3y - 4z = 0 \\ 3x - 7y + 4z = 10 \end{cases}$$

has the same solution as original system.

$\xrightarrow{③ - 3①}$ $\begin{cases} x - 2y + z = 3 \\ 3y - 4z = 0 \\ -y + z = 1 \end{cases}$

Now we have eliminated "x" from ②, ③.

$3 - 8 + 3 = -2$

$\xrightarrow{③ + \frac{1}{3}②}$ $\begin{cases} x - 2y + z = 3 \\ 3y - 4z = 0 \\ -\frac{1}{3}z = 1 \end{cases}$ $\Rightarrow z = -3$

Plug in z, y, $x = 3 + 2y - z$

Plug in $z = -3$, $y = \frac{4}{3}z = -4$.

$\frac{1}{3}②$: $\begin{array}{r} y - \frac{4}{3}z = 0 \\ +)\ -y + z = 1 \\ \hline -\frac{1}{3}z = 1 \end{array}$

$x = -2, \ y = -4, \ z = -3.$

Remark (RK.): (1) This process is called "Gaussian Elimination".

(2) $\begin{cases} ① \\ ② \\ \vdots \\ ⓝ \end{cases}$

Use ①, to eliminate 1st variable from ② ... ⓝ

Use ②, " 2nd " " ③ ... ⓝ

...

$\xrightarrow[\text{Elimination}]{\text{Gaussian}}$ get a upper triangular system.

We solve it from bottom up, called

" back substitution ".

1.2 Matrices and Vectors and Basic Operations

- A **matrix** is simply a rectangle array of numbers, such as,

$$\begin{bmatrix} 1 & 0.7 & 10 \\ \pi & 6 & 0 \end{bmatrix}_{2\times 3}, \quad \begin{bmatrix} \cos(1) & 1 \\ 4 & 6 \\ -10 & e^2 \end{bmatrix}_{3\times 2}$$

number of column

The 1st matrix above is a $\underbrace{2}_{\text{number of row}} \times \overset{\frown}{3}$ matrix and 2nd matrix above is a 3×2 matrix.

- Generally, an $m \times n$ matrix A is a two-dimensional array of $m \cdot n$ numbers:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where m is the number of rows and n is the number of columns. The element a_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$, is called the **entry** of A.
i row, j column.

- A column vector is a matrix where $n = 1$:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}$$

- A row vector is a matrix where $m = 1$:

$$\mathbf{w} = (w_1 \, w_2 \, \cdots \, w_n)$$

§ Three basic operations:

1. Matrix addition:

the same size,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

2. Scalar multiplication: If c is a number, we can multiply a matrix by c:

$$c \times \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} c\cdot a_{11} & c\cdot a_{12} & \cdots & c\cdot a_{1n} \\ c\cdot a_{21} & c\cdot a_{22} & \cdots & c\cdot a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c\cdot a_{m1} & c\cdot a_{m2} & \cdots & c\cdot a_{mn} \end{pmatrix}$$

$$7\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 7 & 14 & 21 \\ 28 & 35 & 42 \end{pmatrix}_{2\times 3}$$

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3. Matrix multiplication:

$$\underbrace{(v_1 \cdots v_p)}_{1\times p}\underbrace{\begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix}}_{p\times 1} = v_1w_1 + v_2w_2 + \cdots + v_pw_p$$

Generally, if $A = (a_{ij})$ is $m \times n$ matrix and $B = (b_{ij})$ is $n \times p$ matrix, then their product $C = AB$ is $m \times p$ matrix and has entries:

$$c_{ij} = (i^{\text{th}} \text{ row of } A) \times (j^{\text{th}} \text{column of } B)$$

$$C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A}{}_{m\times n} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}}_{B}{}_{n\times p}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \\ & & \\ & & \end{pmatrix}$$

where $c_{11} = (a_{11} \;\; a_{12} \;\; \cdots \;\; a_{1n}) \begin{pmatrix} b_{11} \\ \vdots \\ b_{n1} \end{pmatrix} = a_{11}b_{11} + \ldots + a_{1n}b_{n1}.$

* the number of columns of A = the number of rows of B.

Remark:

- Matrix multiplication is associative: $(AB)C = A(BC)$
- Not commutative: in general, $AB \neq BA$.

Example: $A = (1, 2, 3)$ and
$$B = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$
Compute AB and BA.

$$AB = (1\ 2\ 3)\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = (4 + 10 + 18) = (32).$$

$$BA = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}(1\ 2\ 3) = \begin{pmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{pmatrix}_{3\times 3}$$

§ Vectors and matrices provide a convenient notation for linear systems.

- For example, the linear system

$$\begin{aligned} 5x - 4y + 3z &= 19 \\ 4x - 5y + 3z &= 17 \\ x - y - 2z &= 4 \end{aligned}$$

is equivalent to the equation:

$$\begin{pmatrix} 5 & -4 & 3 \\ 4 & -5 & 3 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 19 \\ 17 \\ 4 \end{pmatrix}$$

- In more compact notation, we can write:

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{pmatrix} 5 & -4 & 3 \\ 4 & -5 & 3 \\ 1 & -1 & -2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 19 \\ 17 \\ 4 \end{pmatrix}$$

§ Some special matrices we will see and utilize many times in this course.

- The n-by-n *identity matrix*, typically denoted I or I_n, defined by:

$$I = I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{n\times n}$$

  In other words, I has 1's on the main diagonal, and the off-diagonal elements are 0. It's easy to check that

$$I_n A = A \ \text{ and } \ BI_n = B$$

  for any matrix A with n rows and any matrix B with n columns.

- The m-by-n zero matrix, typically denoted O or $O_{m\times n}$, which has all zero entries. It's easy to check that

$$O_{m\times n}A = O_{m\times k}$$

  for any n-by-k matrix A, and

$$BO_{m\times n} = O_{k\times n}$$

  for any k-by-m matrix B.

§ Some useful notations:

- if $a_1, a_2, \ldots, a_n$ are n numbers, we will denote by $\operatorname{diag}(a_1, \ldots, a_n)$ the following n-by-n matrix:

$$\operatorname{diag}(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_n \end{pmatrix}$$

In other words, $\mathrm{diag}(a_1, \ldots, a_n)$ has $a_1, \ldots, a_n$ on the main diagonal, and the off-diagonal elements are 0.

- In this notation, $I_n = \mathrm{diag}\underbrace{(1, \ldots, 1)}_{n \text{ times}}$

§ The **augmented matrix** for a linear system appends the right hand side as an extra column to the coefficient matrix.

- For example, the augmented matrix for the linear system

$$\begin{aligned} x + 2y + 2z &= 2 \\ 2x + 6y &= 1 \\ 4x + 4z &= 0 \end{aligned}$$

is the 3-by-4 matrix:

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 6 & 0 & 1 \\ 4 & 0 & 4 & 0 \end{pmatrix}$$

- For clarity, this may also be written like this:

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 2 & 6 & 0 & 1 \\ 4 & 0 & 4 & 0 \end{array}\right)$$

- Gaussian elimination can be expressed entirely in terms of the augmented matrix.

- Also, the operations of Gaussian elimination can be used to update the augmented matrix.

Example:

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 2 & 6 & 0 & 1 \\ 4 & 0 & 4 & 0 \end{array}\right) \quad \text{is identical to} \quad \begin{cases} x + 2y + 2z &= 2 \quad - 1 \\ 2x + 6y &= 1 \quad - 2 \\ 4x + 4z &= 0 \quad - 3 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 2 & -4 & -3 \\ 0 & -8 & -4 & -8 \end{array}\right) \xrightarrow[\text{(eliminate "x") in 2, 3}]{2 - 2\,1 \quad 3 - 4\,1} \begin{cases} x + 2y + 2z = 2 \\ 2y - 4z = -3 \\ -8y - 4z = -8. \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 2 & -4 & -3 \\ 0 & 0 & -20 & -20 \end{array}\right) \xrightarrow[\text{(eliminate "y" in 3)}]{3 + 4\,2} \begin{cases} x + 2y + 2z = 2 \\ 2y - 4z = -3 \\ -20z = -20 \end{cases}$$

Using "back-substitution",

$\underline{z = 1}$, 2 $2y = -3 + 4z = -3 + 4 = 1$, $y = \underline{1/2}$.

Plug $z = 1$, $y = 1/2$ into 1, $x = 2 - 2y - 2z$

$= 2 - 1 - 2 = \underline{-1}$

Ans: $x = -1$, $y = 1/2$, $z = 1$ #

Remark:

- Adding/subtracting a multiple of one row to/from another row is called an **elementary row operation**.
- Each elementary row operation is associated with an **elementary matrix**, defined by applying the elementary row operation to the identity matrix.

Example. The elementary matrix associated with adding 3 times the 3rd row to the 1st row is:

$$E = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Multiplying a matrix A on the left by an elementary matrix E performs the associated row operation on A. For example, check that:

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a+3g & b+3h & c+3i \\ d & e & f \\ g & h & i \end{pmatrix}$$